AA2. Lecture 30.

04. 04. 2022.

Last time:

- Direct sums/summands of s.s. algebras are s.s.

  (as are hom. images, iso copies, and quotients of s.s algebras) } new from old

- Def. of the Jacobson radical: $J(A) := \bigcap_{\substack{M \text{ is a max.} \\ \text{left ideal of } A}} M$.
  (§4.3)

- Properties of $J(A)$.

  In particular, for an algebra A of finite length, we have

  - A is s.s $\iff J(A) = 0$, · an A-module V is s.s $\iff J(A).V = 0$.

Today:

- Proof of the properties.

  The statements are worth repeating ...

Thm 4.23. Suppose that A is an algebra of finite length (i.e., A has a comp series). Let $J = J(A)$. Then the following holds.

(a) J is the intersection of finitely many maximal left ideals.

(b) $J = \bigcap_{S \text{ simples}} \mathrm{Ann}(S)$. → In particular, J kills all simples

(c)✓ J is a two-sided ideal.

> **Note:** Recall that $\mathrm{Ann}(V)$ is a two-sided ideal of A for every A-module V. So (b) $\Rightarrow$ (c).

(d) J is a nilpotent ideal. Moreover, if n is the length of A, then $J^n = 0$.

(d)' (Not in the book but true) J contains every nilp. left ideal of A.

J is the largest nilpotent left ideal.

(e) A/J is s.s. } J is the smallest 2-sided ideal whose corr. quotient is s.s.

(f) If $I \subsetneq A$ is a two-sided ideal s.t. A/I is s.s., then $J \subseteq I$.

(g) A is s.s iff $J = 0$ → J detects s.s. of A.

(h) An A-module V is s.s iff $J \cdot V = 0$ → J detects s.s. of A-modules.

Pf: (a) (finite intersection) If not, we can find an inf. chain of ideals

$$(*)\qquad (A \supseteq)\ M_1 \supsetneq M_1 \cap M_2 \supsetneq M_1 \cap M_2 \cap M_3 \supsetneq \cdots$$

where the components are the intersection of left max. ideals $M_1, M_2, M_3, \cdots$.

Write $I_j = M_1 \cap M_2 \cap \cdots \cap M_j$. Then we have I_j / I_{j+1} is simple $\forall j \geq 1$. (Ex. prove the claim w/ the 2nd Iso Thm.) This cannot happen since A has finite length.

(b) $J = \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$.

We first prove $J \supseteq \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$. Take $a \in \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$. Then a annihilates every simple simple module of A. To prove $a \in J$, we prove that a lies in every maximal left ideal of A. Let M be a maximal left ideal. Then the quotient module A/M is simple.

So $a \cdot A/M = 0$. In particular $a \cdot (1_A + M) = a + M = 0_{A/M}$, so $a \in M$. as desired.

Next, we prove that $J \subseteq \bigcap_{S \text{ simple}} \mathrm{Ann}(S)$. It suffizes to show that for every simple module S of A, $J \subseteq \mathrm{Ann}(S)$, i.e., $JS = 0$. We prove this by contradiction:

if $JS \neq 0$, then for some nonzero $s' \in S$, we must have $J \cdot s' \neq 0$.

Note that $J \cdot s'$ is a submodule of S (key: $A \cdot (J \cdot s') \subseteq (A \cdot J) \cdot s' \subseteq J \cdot s'$),

it follows that $J \cdot s' = S$ (since S is simple and $J \cdot s' \neq 0$).

In particular, $\exists j \in J$ s.t $j \cdot s' = s'$. But then $(j-1) \cdot s = j \cdot s - 1 \cdot s = j \cdot s - s = 0$.

so $\underline{j-1 \in \mathrm{Ann}(s')} := \{ x \in A : x \cdot s' = 0 \}$. But $\underline{j \in \mathrm{Ann}(s')}$ since $\mathrm{Ann}(s')$ is maximal.
(because $A/\mathrm{Ann}(s') \cong S$)

so $1 = j - (j-1) \in \mathrm{Ann}(s')$, i.e., $1 \cdot s' = 0$. This cannot happen since

$1 \cdot s' = s' \neq 0$ by module axioms.

It follows that (b) holds. By earlier discussion, (c) must also hold (J is a 2-sided ideal).

(d). (nilpotency) strategy: hit the comp. factors of A with J.

Suppose length$(A)=n$. Take a comp. series $0=V_0 \subset V_1 \subset \cdots \subset V_n = A$ of A.

Then V_i/V_{i-1} is simple for all $1 \leq i \leq n$. So by (b) we have $J \cdot V_i/V_{i-1} = 0$.

$\left(\forall j \in J,\ a \in A \quad j : a + V_{i-1} \mapsto j \cdot a + V_{i-1} \overset{\shortparallel}{0} \right.$

$\left. \text{i.e., } J \cdot V_i \subseteq V_{i-1} \right)$

In other words, $J \cdot V_i \subseteq V_{i-1}$.

It follows that $J^n = \underbrace{J^n \cdot 1_A}_{\{x \cdot 1_A \,:\, x \in J^n\}} \in V_0 = 0$.

$\in A = V_n$

(e) – h): next time.