AA2. Lecture 3.

01. 14. 2022.

Last time:

- isomorphism theorems for gps
- def of rings and fields

commutative ring where every nonzero elt is invertible

E.x. Are the following rings fields?

$(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$

$\times$ (e.g. 2 has no inverse) $\checkmark$ $\left(\frac{m}{n} \mapsto \frac{n}{m}\right)$ $\checkmark$ $\left(x \mapsto \frac{1}{x}, \text{ inverse } \forall x \neq 0\right)$

$(M_n(\mathbb{R}), +, \cdot)$, $n > 1$. $\times$ not commutative; also, not all nonzero matrices are invertible, e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Today:

- Vector spaces & linear algebra
- Def. of algebras over fields.

Let k be a field for the rest of today.

1. Vector spaces

Def: A vector space (v.s.) over k is a triple $(V, +, \cdot)$

where

- $(V, +)$ is an abelian gp $\rightarrow$ expands to five conditions. e.g. $v+w = w+v \ \forall v, w \in V$.
- $\cdot$ is a map $\cdot : k \times V \to V$, called scaling with the following properties:

  (a) $1 \cdot u = u$

  (b) $c \cdot (u+v) = c \cdot u + c \cdot v$

  (c) $(c+d) \cdot u = c \cdot u + d \cdot u$

  (d) $(c \cdot d) \cdot u = c \cdot (d \cdot u)$

  $\forall c, d \in k,\ u, v \in V$.

  "the action behaves well".

We assume familiarity with basic linear algebras: vectors, matrices, determinants, linear maps, bases, etc.

We should also be comfortable with abstract vector spaces (outside $\mathbb{R}^n$, e.g. some spaces of functions or maps).

E.g. (1) $(\mathbb{R}^n, +)$ forms a v.s. over $\mathbb{R}$.

e.g. $n=3$. $\quad 2 \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$

(2) $(k^n, +)$ is a v.s. over k.

$\rightarrow$ E.x: check the v.s. axioms.

Note: $(\mathbb{C}, +)$ is also a v.s. over $\mathbb{C}$. As a $\mathbb{C}$-v.s., $\mathbb{C}$ has a basis $\{1\}$ and dim 1.

(3) $(\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}, +)$ is a v.s. over $\mathbb{R}$.

$\cdot : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$.

e.g. $\pi \cdot (2+3i) = 2\pi + (3\pi) i$

As an $\mathbb{R}$-vector space, $\mathbb{C}$ has a basis $\{1, i\}$ and hence has dimension 2.

14) $M_n(k)$, the set of all $n\times n$ matrices with entries from k, forms a v.s. over k.

e.g. $M_2(\mathbb{R})$. typical elts: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$.

$+$: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 7 \end{bmatrix}$

scaler mult: $7 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}$

By properties of matrix operations (just scaling and addition), we know that $M_n(\mathbb{R})$ is a vector space,

Note: So far we haven't considered mult. of matrices in the discussion at all. The mult. and its nice properties will make $M_n(k)$ more than a v.s: it will make $M_n(k)$ an "algebra over k" (to be defined).

2. Basic notions of linear algebra

Def. · A <u>linear map</u> between two k-v.s. V and W is a map

$$\varphi: V \to W \quad \text{s.t.} \quad \begin{cases} (a) & f(u+v) = f(u) + f(v) \quad \forall u, v \in V \\ (b) & f(c \cdot u) = c \cdot f(u) \quad \forall c \in k,\ u \in V. \end{cases}$$

Note: Recall that (a), (b) imply that $f(0_V) = 0_W$:

$$\underline{f(0_V)} = f(0_V + 0_V) = \underline{f(0_V) + f(0_V)}$$

Adding the additive inverse of $f(0_V)$ to both sides yields $0_W = f(0_V)$.

· A basis of a v.s. V is a subset $B \subseteq V$ s.t.

$$\begin{cases} B \text{ is } \underline{\text{linearly independent}} : \left(\sum_{i=1}^{n} c_i v_i = 0,\ v_1, \dots, v_n \in B,\ c_1, \dots, c_k \in k \Rightarrow c_i = 0\ \forall i \right) \\ B \text{ spans } V, \text{ i.e., every elt of } V \text{ is a lin comb. of elts of } B. \end{cases}$$

· Dimension: **Fact:** Every v.s. has at least one basis, and all bases of it have the same size.

Def: That size is called the dimension of V. Notation: $\dim_k(V)$.

E.g. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ are both bases of $\mathbb{R}^2$.

and $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$

E.g. Earlier we saw that $\dim_{\mathbb{R}}(\mathbb{C}) = 2$

and $\dim_{\mathbb{C}}(\mathbb{C}) = 1$.

· A v.s. V over k is finite dimensional if it has a finite spanning set.

↳ implies that V has a finite basis.

Next time: · more on bases

· def. of k-algebras.