AA2. Lecture 29.

Last time:

- HW
- Semisimple algebras:

Prop: Hom. images of s.s. algebras are semisimple.

Cor: Iso copies and quotients of s.s. algebras are s.s.

Also: Subalgebras of s.s. algebras may not be s.s.

$T_n(k) \subseteq M_n(k)$.

Today:

- direct sum/summands of s.s. algebras are s.s.
- Jacobson Radicals: def. and statement of properties

1. Direct sum / summands

Prop: Let $A_1, A_2, \ldots, A_r$ be finitely many k-algebras. Let $A = A_1 \times A_2 \times \cdots \times A_r$.

Then A is a s.s. algebra iff A_i is a s.s. algebra for all $1 \le i \le r$.

Pf of the "only if" implication: (A is s.s $\Rightarrow$ A_i is s.s $\forall i$)

Suppose A is s.s. For each i, we can consider the projection hom

$$\pi_i : A = A_1 \times \cdots \times A_r \longrightarrow A_i$$
$$(a_1, a_2, \ldots, a_r) \longmapsto a_i .$$

We have $A_i = \operatorname{Im}(\pi_i)$, and hom. images of s.s. algebras are s.s,

so A_i is s.s. $\square$

Pf of the "if" implication :

Preparation : Recall (L15) that for any algebra A and any two-sided ideal I of A, there is a bijection

$$\{\text{modules of } A/I\} \underset{}{\overset{\text{inflation}}{\rightleftarrows}} \left\{\begin{array}{c}\text{modules of } A \\ \text{annihilated by } I\end{array}\right\}$$

Thm (Thm 4.17) : The above bijections preserve s.s. semisimplicity,

i.e. they restrict to bijections

$$\{\text{s.s. module of } A/I\} \rightleftarrows \left\{\begin{array}{c}\text{s.s. module of } A \\ \text{annihilated by } I\end{array}\right\}.$$

pf: E.X.

Pf ("if"): Suppose A_i is s.s for all $1 \le i \le r$. We will show that A is a s.s. algebra by showing that every A-module M is s.s. Note that

(1) We have a decomposition of A-modules $M = \bigoplus_{i=1}^{r} \varepsilon_i M$ where $\varepsilon_i = (0, 0, \cdots, \underset{\substack{\downarrow \\ \text{ith spot}}}{1_{A_i}}, 0, \cdots, 0)$. (Lemma 3.30)

Consequently, to show M is a s.s. it suffices to show that $M_i := \varepsilon_i M$ is a s.s. A-module $\forall i$.

(2) In the above notation, each M_i is an A_j-module $\forall j$, with action

$$A_j \times M_i = A_j \times (\varepsilon_i M) \to M_i, \quad a_j \cdot (\varepsilon_i m) \to (a_j \underbrace{\varepsilon_j \varepsilon_i}_{0 \text{ if } i \neq j}) \cdot m.$$

(The action corresponds to the rep $A_j \to A \xrightarrow{\text{the } A\text{-rep}} \operatorname{End}(M_i)$.

$$a_j \mapsto (0, \cdots, \underset{\substack{\text{jth} \\ \downarrow}}{a_j}, 0, \cdots 0) = a_j \varepsilon_j \nearrow$$
)

We need to show that each M_i is a s.s. A-module.

Since A_i is a s.s. algebra, M_i is a s.s. A_i-module. therefore

M_i is a s.s. A-module by Thm 4.1]. $\square$

$\left(A_i = \mathrm{Im}(\pi_i) = A/\ker(\pi_i) \right)$

Eg: Since $M_n(k)$ is a s.s. algebra for each n, and same for $M_n(D)$ for any division algebra D over k, it follows that every alg. of the form

$$(*) \qquad M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k)$$

is s.s. In Ch. 5, we'll see that any s.s. algebra over k has to be of the form $(*)$ $\rightarrow$ the Artin-Wedderburn Thm.

2. Jacobson Radicals

Let A be a k-algebre.

Def: The Jacobson radical $J(A)$ of A is the intersection of all maximal left ideals of A.

Note:
- $J(A)$ will be a measure of "how far A is from being s.s".
- $J(A)$ is not called the "left Jacobson radical" ... The reason is that in fact, $J(A)$ must also equal the intersection of all maximal right ideals of A.

Def: (Note that given any two left ideals I, J of A, the set $IJ = \text{Span}\{xy \mid x \in I, y \in J\}$ is again a left ideal; in particular, we have a chain of ideals $I \supseteq I^2 \supseteq I^3 \supseteq I^4 \supseteq \cdots\cdots$.)

We say an ideal $I \subseteq A$ is nilpotent if $I^r = 0$ for some $r \geq 1$.

Thm. (Properties of $J(A)$) Suppose A is of finite length (as a regular module)

Let $J = J(A)$. Then the following holds.

- J is the intersection of finitely many maximal left ideals.

- $J = \bigcap_{S \text{ is a simple } A\text{-module}} \mathrm{Ann}_A(S)$ $\qquad$ $\mathrm{Ann}_A(S) = \{a \in A : a.s = 0 \ \forall s \in S\}$.

* · J is a nilpotent ideal.

* · A/J is s.s.

* · A is s.s. iff $J = 0$.

... and more.

Next time: more properties of $J(A)$.

proofs