AA2. Lecture 28. 03. 30. 2022.

Last time:

- properties of s.s. modules, new from old
- Thm: An algebra is a s.s. algebra iff all its modules are s.s.

Today:

- Semisimple algebras
  - def: regular module is s.s.
  - thm: all modules s.s. → often very useful
- HW questions.

1. HW.

4.5. $G = S_3$, $A = kS_3 \curvearrowright V = k^3 = \mathrm{Span}\{v_1, v_2, v_3\}$ by permutation

$$U = \mathrm{Span}\{v_1 + v_2 + v_3\} \subseteq V \to \text{submodule}.$$

(b). char $k \neq 3$. show that V is a s.s. module by decomposing it into simples.

Do $V = U \oplus W$ where $W = \{ av_1 + bv_2 + cv_3 \mid a+b+c=0 \}$

$$= \mathrm{Span}\{v_1 - v_2, v_2 - v_3\}.$$

Need: · W is a submodule.

· $V = U \oplus W$, i.e., $V = U + W$ and $U \cap W = 0$

$$xv_1 + yv_2 + zv_3 = \underbrace{\left(\frac{x+y+z}{3}\right)(v_1+v_2+v_3)}_{\in U} + \underbrace{\left(\frac{2x-y-z}{3}v_1 + \frac{2y-x-z}{3}v_2 + \frac{2z-x-y}{3}v_3\right)}_{\in W}.$$

(c) $\operatorname{char} k = 3$ Show that V is not s.s.

Can't divide by 3.

$v_1 + v_2 + v_3 \in W$ since $1+1+1 = 3 = 0$, so $U \cap W \neq 0$; rather, $U \subseteq W$.

Pf strategy: If V were s.s, then W must be s.s, so U must have a complement in W. Show that such a complement cannot exist.

c.r.

(from the book P.282.)

Evan: Suppose Y is a complement of U in V, so that $V = U \oplus Y$.

Take $0 \neq v := (a, b, c) \in Y$. Then $v + (123)\cdot v + (132)\cdot v = (a+b+c)\cdot(1,1,1) \in Y \cap U$.

So $a+b+c = 0$. $\Rightarrow Y \subseteq W := \{(a,b,c) \mid a+b+c = 0\}$.

Show this cannot happen by dimension considerations and the fact that $U \cap W \neq 0$.

2. Semisimple algebras

Warm-up Exercise: Lemma 3.5. Let A be a k-algebra, and let $B = A/I$ where I is a proper two-sided ideal of A. If S is a simple B-module, then S is a simple A-module under the usual inflated action $a \cdot s = (a+I) \cdot s$.

Pf: Ex.; think/use the cyclicity test for simplicity.

Recall: More generally, if $\varphi : A \to B$ is an algebra hom, then any B module V is automatically an A-module by inflation, under the action

$$a \cdot v = \varphi(a) \cdot v.$$

New semisimple algebras from old.

Prop 1: (homomorphic images) Let $\varphi: A \to B$ be a surjective algebra hom.

If A is s.s, then B is s.s.

Pf: We'll show that B is s.s. by showing that every B-module V is s.s.
So take V a B-module. Inflate V to an A-module. Since A is s.s.,
V is a s.s. A-module, so $V \overset{A}{=} \sum_{i \in I} S_i$, as A-modules, for some simple modules
$\{S_i : i \in I\}$. We will show that each S_i is automatically a B-module
and a simple B-module, so that we also have a equalities of B-modules

$$V \overset{B}{=} \sum_{i \in I} S_i$$

and we'll be done.

S_i is a B-module: To show S_i is a B-module, note that

(1) S_i is a subspace of V since S_i is a A-submodule of V.

(2) $\forall b \in B$, $s \in S_i$, $b \cdot S_i = \varphi(a) \cdot s \in A \cdot S_i \subseteq S_i$ for any $a \in A$ s.t. $\varphi(a) = b$.

where such an elt $a \in A$ exists since φ is surj.

By (1), (2), we have S_i is a B-module.

S_i is a simple B module: Let U be a B submodule of S_i. Then U is an A-module by inflation. Since S_i is simple as an A-module, it follows that U, a submodule of S_i, is either 0 or S_i. Thus, S_i is simple as a B-module. $\square$

Corollary 1: (Iso Copies) If A and B are iso algebras, then A is s.s. iff B is s.s.

Pf: Take an iso $\varphi: A \to B$ and also consider the inverse iso $\varphi^{-1}: B \to A$. Both φ and φ^{-1} are surjective, so the conclusion follows from the prop.

Corollary 2: (Quotients) Let A be a s.s. algebra and $I \subseteq A$ a two-sided ideal of A. Then A/I is s.s.

Pf: ("quotients $\equiv$ hom. images") Apply the prop. to the projection

$$A \to A/I, \quad a \mapsto a+I.$$

Next time: · more properties of s.s. algebra.