AA2. lecture 27. 03. 28. 2022.

Last time:

- Examples of non-semisimple modules:

$$T_n(k) \curvearrowright k^n, \quad k[x] \curvearrowright k^2 \ \left(x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right), \quad A := k[x]/\langle x^t \rangle \curvearrowright A \ \text{if } t > 1.$$

- TFAE for an A-module M:

(1) M is s.s. i.e., a direct sum of simples

(2) M is completely reducible, i.e., every submodule U of M has a complement. $M = U \oplus V$.

(3) M is a sum of simples.

- Properties of s.s. modules ...

Corollary 1: A submodule of a s.s module M is always iso. to a quotient of M, and vice versa.

Corollary 2: Homomorphic images of s.s. modules are s.s.

Today:

- more corollaries / properties of s.s. modules.

- proof of the theorem that

  an algebra is s.s iff every module of it is s.s.

1. Properties of s.s. modules.

Corollary 3: (Iso.s preserve s.s. modules)

Suppose $\varphi: V \to W$ is an isomorphism of A-modules. Then V is s.s iff W is s.s.

Pf: V is s.s $\Rightarrow \operatorname{Im}\varphi$ is s.s by Cr. 2 $\xRightarrow[\text{since } \varphi \text{ is surj}]{\operatorname{Im}\varphi = W}$ W is s.s.

Similarly, φ^{-1} is an isomorphism from W to V, so

$$W \text{ is s.s.} \Longrightarrow \operatorname{Im}\varphi^{-1} = V \text{ is s.s.}$$

$\square$

Corollary 4. (submodules / quotients of s.s. modules are s.s).

Let V be a s.s. A-module. Then every submodule of V is s.s and every quotient of V is s.s.

Pf: (1) Every quotient of V is of the form V/U for a submodule U of V, which equals the hom. image of the hom $\varphi: V \to V/U$,

"quotients $\equiv$ hom images".

$x \mapsto x+U$ $\forall x \in V$. Thus, the quotient V/U is s.s. by Corollary 2.

(2) By Corollary 1, it follows that all submodules of V are s.s. as well

Note: We already showed that a subalgebra of a s.s. algebra may not be s.s, which should be contrasted with (2).

Corollary 5. (Direct sum / summands of s.s. modules are s.s.)

Let $(V_i)_{i\in I}$ be a family of nonzero A-modules. Then $V := \bigoplus_{i\in I} V_i$ is s.s

iff V_i is s.s. for all $i \in I$.

Pf: Recall that there is a natural (injective) inclusion hom $\iota_i : V_i \to V$

$x_i \mapsto (\dots, 0, x_i, 0, 0, \dots)$

$\downarrow$ ith spot

with $\operatorname{Im} \iota_i$ being a submodule of V iso. to V_i.

"only if": V is s.s. $\Rightarrow$ $\operatorname{Im}\iota_i$ is s.s by Cor 4 $\forall i \in I$ $\Rightarrow$ V_i is s.s by Cor. 3. $\forall i\in I$.

"if": V_i is s.s $\forall i \in I$ $\Rightarrow$ $\forall i \in I$, $V_i = \bigoplus_{j\in J_i} S_{ij}$ for simples S_{ij}, so

$$V = \bigoplus_{i\in I} \iota_i(V_i) = \bigoplus_{i\in I} \iota_i\left(\bigoplus S_{ij}\right) = \sum_{i\in I, j\in J_i} \underbrace{\iota_i(S_{ij})}_{\text{simple or zero}} \text{ is semisimple.} \quad \square$$

since $\iota_i(S_{ij}) \cong S_{ij} / \underbrace{\ker \iota_i}_{0 \text{ or } S_{ij}}$

Summary: Hom images, iso copies, quotients, submodules, direct sums, and direct summands of s.s modules are all s.s.

2. All modules of a s.s. algebra are s.s

Thm. An k-algebra A is a s.s. algebra (def: the regular module $A \circlearrowright A$ is s.s.) iff every module of A is s.s.

Pf: "if": Every A-module is s.s. $\Rightarrow$ the reg. module is s.s $\Rightarrow$ A is s.s.

"only if": Suppose A is a s.s. algebra. Let V be an arbitrary A-module. We need to show that V is a s.s. module. To do so, pick a basis $B = \{v_i \mid i \in I\}$ of V.

Consider the map $\psi : \bigoplus_{i\in I} A \longrightarrow V$, $(a_i)_{i\in I} \longmapsto \sum_{i\in I} a_i v_i$.

The above sum is finite since only finitely many a_i's can be nonzero.

and ψ is surjective since $B = \{v_i : i \in I\}$ is a basis of V. So

$V = \operatorname{Im}\psi$. Moreover, ψ is a hom of A-modules by straightforward proofs.

Ex. E.g. $\psi((a_i)+(b_i)) = \psi((a_i+b_i)) = \sum (a_i+b_i) v_i$

$$= \sum a_i v_i + \sum b_i v_i$$
$$= \psi((a_i)) + \psi((b_i))$$

Now, since A is s.s, the domain $\bigoplus_{i\in I} A$ is s.s by Corollary 5,

so $V = \operatorname{Im}\psi$ is s.s. by Cor. 2. We are done. $\square$

Note: The argument above really says "every A module is a hom image/quotient of a direct sum of copies of the regular module".

Next time: s. s. algebras.