AA2. Lecture 26

03.18.2022.

Last time:

- Schur's Lemma and its applications

- Preview on semisimple modules and algebras

  Examples of s.s. modules: simples, k-modules, $M_n(k) \circlearrowright M_n(k)$

  $\Downarrow$

  $M_n(k)$ is a s.s. algebra

Today:

- Two non-semisimple modules

- Equivalence defs and corollaries of the equivalences

1. Nonexamples

(a) $A = T_n(k) \curvearrowright V = k^n$. Recall that there are exactly $n+1$ submodules of V

$$\begin{bmatrix} * & * & * \\ 0 & \ddots & \\ & & * \end{bmatrix}$$

and they form a chain

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V = k^n$$

where $V_i = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ 0 \\ 0 \\ \vdots \end{bmatrix} \right\} \quad \forall i \quad (V_0 = 0)$

Thus, no two nonzero submodules of V intersect trivially (they both contain V_1), therefore we cannot write V as a direct sum of simple submodules, so V is not s.s.

Note: · We shall see that the fact that V is not a s.s. module implies that $T_n(k)$ is not a s.s. algebra. Thus, $T_n(k)$ is an example of a subalgebra of a s.s. algebra ($T_n(k) \subseteq M_n(k)$) that is s.s.

(b) From Lec: Recall from HW7 that the $k[x]$-module $k[x] \circlearrowright V = k^2$, $x \cdot - = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is indecomposable but not simple. Moreover, any nonzero submodule of V must contain $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, no two nonzero submodules of V can intersect trivially, so V must be non-s.s. $\square$

b) $A = k[x]/\langle x^t \rangle$ $(t \geq 1)$. Consider the regular module $A \curvearrowright A$.

Recall that the only simple submodule of $A \curvearrowright A$ is $V = k[x]/\langle x \rangle$, where x is the only irr. factor of x^t. On V, the elt $x + \langle x^t \rangle$ acts as the scalar 0:

$$\left(x + \langle x^t \rangle\right) \cdot \left(f + \langle x \rangle\right) = xf + \langle x \rangle = 0$$

By preservation of scaling actions, $x + \langle x^t \rangle$ must act as 0 on all simple submodules of V. It follows that $x + \langle x^t \rangle$ must act as 0 on all s.s. modules of A. It further follows that the regular module A is s.s. iff $t = 1$.

direct sum of simples

2. Equivalent definitions for s.s. modules

Thm (Thm 4.3): Let A be a k-alg. and V a nonzero A-module. TFAE:

(1) (def.) V is s.s., i.e., V is a direct sum of simple submodules.

(2) (complete reducibility) V is completely reducible in the sense that for any submodule U of V, there is a submodule C of V s.t. $V = U \oplus \underline{\underline{C}}$.

"complement of U"

(3) (sum) V is a sum of simple submodules, i.e., $V = \sum S_i$ for some index set I, where S_i is simple for all $i \in I$.

Rmk: Clearly (1) $\Rightarrow$ (3), but we'll skip the full proof for now. (It's in the book).
We'll derive some useful conseq. of the equivalence instead.

3. Properties of s.s. modules / Consequences of Thm 4.3

Let A be a k-algebra and V an A-module

Corollary 1. (submodules $\equiv$ quotients for s.s. V) If V is s.s., then every submodule of V is iso. to a quotient module of V, and vice versa.

Pf. We use the complete reducibility characterization of s.s:

(1) Let U be a submodule of V. Since V is comp. red., we have $V = U \oplus C$. But then we have $U \cong V/C$ $\left(V/C = U \oplus C / C = U + C / C \cong U / U \cap C = U/0 \cong U \right)$.

(2) Let U be a quotient of V, say $U = V/W$ for some submodule W of V.

Since W is comp. red., we have $V = W \oplus C$ for some submodule C of V.

whence $U = V/W \cong C$. $\square$

Corollary 2. (Homomorphic images of s.s modules are s.s.)

Let $\varphi : V \to W$ be a nonzero A-mod hom. If V is s.s, then $\mathrm{Im}\,\varphi$ is s.s.
In particular, if φ is surj, then W is s.s.

Pf: We use the sum char. of s.s. and the following fact:

> "if $\varphi : S \to W$ is an A-mod hom and S is a simple A-module, then either $\varphi = 0$ or $\mathrm{Im}\,\varphi$ is simple and iso to S."
>
> **Pf:** Suppose $\varphi \neq 0$. Then $\ker\varphi \neq S$, so $\ker\varphi = 0$, so $\mathrm{Im}\,\varphi \cong S/\ker\varphi = S/0 \cong S$.

Suppose V is s.s.. Then $V = \sum_{i \in I} S_i$ for simples S_i ($i \in I$), so

$$\mathrm{Im}\,\varphi = \varphi(V) = \varphi\Big(\sum_{i\in I} S_i\Big) = \sum_{i \in I} \underbrace{\varphi(S_i)}_{\text{simple or zero}} .$$

Since each $\varphi(S_i)$ is simple or zero, and we can discard the zeros

it follows that $\mathrm{Im}\,\varphi$ is a sum of simples, so $\mathrm{Im}\,\varphi$ is s.s. □

Next time: more corollaries on properties of s.s. modules.

Upshot: Homomorphic images, Isomorphic copies, Submodules, quotient modules, direct sums, and direct summands of s.s. modules are s.s.