AA2. Lecture 25.

03.06.2022.

Last time:

- proof of the Jordan–Hölder thm
- properties of module lengths: if $U \leq V$, then

  $\ell(V) = \ell(U) + \ell(V/U)$, and $\ell(U) \leq \ell(V)$, with equality holding iff $U = V$.

Today:

- Schur's Lemma and its applications (last topic from Ch. 3)
- Overview of semisimplicity results. (Ch. 4)

1. Schur's Lemma

Prop. (Schur's Lemma) Let A be a k-alg. and let S, T be simple modules of A.

(a) Every A-module hom $\phi : S \to T$ is either 0 or an isomorphism.
invertible

(**Rephrase:** There is no nonzero hom. between two noniso. simples.)

In particular, $\mathrm{End}_A(S) = \mathrm{Hom}(S, S)$ is a division algebra.

(b) If S is f.d. and k is alg. closed, then every $\phi \in \mathrm{End}(S)$ equals $\phi = \lambda\, \mathrm{id}_S$ for some $\lambda \in k$.

pf: (a) Suppose $\phi : S \to T$ is a nonzero hom. Then $\ker\phi \subseteq S$, $\ker\phi \neq S$ and $\mathrm{Im}\,\phi \subseteq T$, $\mathrm{Im}\,\phi \neq 0$.
Since S and T are simple, it follows that $\ker\phi = 0$ and $\mathrm{Im}\,\phi = T$, so that ϕ is inj. and surj. and hence an iso.

(b) Recall the following fact: If k is algebraically closed and V is a f.d. k-v.s., then every k-linear map $f: V \to V$ must have at least one eigenvector.

(think characteristic polynomials)

Now take a nonzero hom $\phi: S \to S$. We need to show that $\phi = \lambda \cdot id_S$ for some $\lambda \in k$. By the above fact, ϕ admits an eigenvector $v \neq 0$, say with e-value λ. Then $\phi(v) = \lambda v$ so that $(\phi - \lambda id)(v) = 0$, i.e., $0 \neq v \in \ker(\phi - \lambda id)$.

Here, $\phi - \lambda id$ is a hom from S to S, and we just showed it has a nontrivial kernel so it's not an iso. By (a), we must have $\phi - \lambda id = 0$, so $\phi = \lambda id$.

The desired claim follows.

Corollary 1: Let k be alg. closed and V a f.d. simple of a k-alg. A.

Recall that the center of A is $Z(A) = \{ a \in A : ab = ba \ \forall b \in A \}$.

For all $a \in Z(A)$, the elt a must act on V as a scaling map, i.e., we must have $a \cdot v = \lambda v \ \forall v \in V$ for some $\lambda \in k$.

$(a \cdot - = \lambda \cdot id_V)$

Pf: By Schur's Lemma (b), it suffices to show that the map $a \cdot - : V \to V, \ v \mapsto a \cdot v$ is an A-module hom from V to V. This map is clearly linear and a gp hom, so it remains to check that it commutes with the module actions. i.e., that

$$a \cdot (b \cdot v) = b \cdot (a \cdot v) \ \forall b \in A.$$

The last equation holds since

$$a \cdot (b \cdot v) = ab \cdot v \overset{*}{=} ba \cdot v = b \cdot (a \cdot v),$$

where $*$ holds since $a \in Z(A)$.

Corollary 2: Let k, A, V be as in Cor. 1.

If A is commutative, then $\dim(V) = 1$.

"Every f.d. simple of a comm. algebra over an alg. closed field has dimension 1."

Pf: Since A is comm, we have $Z(A) = A$, so every elt of A acts by scaling on V. Now take any nonzero elt $v \in V$. Then the submodule

$$Av = \{ a \cdot v \mid a \in A \} = \{ \lambda_a \cdot v \mid a \in A \} \subseteq V$$

must be the span of v. Since V is simple, we must have $V = Av = \operatorname{Span}(v)$,

so $\dim V = 1$. $\square$

2. Overview of semisimple modules and semisimple algebras

Let A be a k-algebra.

Def: (semisimple module) s.s. A <u>semisimple</u> A-module is a nonzero A-module that is a direct sum of simple submodules.

E.g. (1) Simples are s.s. (2). $A = k$, $\Rightarrow$ All A-modules ($= k$-v.s.) are s.s.

V, basis $I \rightarrow$, $V = \bigoplus_{i \in I} V_i$. "$\cong$" $\mathbb{C} = k^n$.

(3) $A = M_n(k) \overset{\text{regular}}{\curvearrowright} V = M_n(k)$ is s.s. because

$V = \bigoplus_{i=1}^{n} C_i$ where $C_i = \left\{ \begin{array}{c} \left[\begin{array}{c|c|c} 0 \cdots & \begin{smallmatrix} * \\ * \\ \vdots \\ * \end{smallmatrix} & 0 \cdots 0 \end{array}\right] \\ i \end{array} \right\}$, which is simple as discussed before.

Def: (semisimple algebra) We say an algebra A is <u>semisimple</u> iff it is semisimple when regarded as a left regular module.

E.g. $M_n(k)$ is a s.s. algebra by the earlier example (3).

Some remarkable facts (sketch)

1. Thm 4.11: An k-algebra A is s.s. iff every A-module is s.s.

   (if: obvious; only if: later)

2. Thm 5.9 (Artin-Wedderburn): Every s.s. algebra is (iso to.) a direct product $M_1 \times M_2 \times \cdots \times M_n$ of matrix algebras.

3. Thm 6.3. (Maschke): Group algebras of finite gps are usually s.s. including whenever $\mathrm{char}(k)=0$

E.g. Fact: The gp alg. $\mathbb{C}S_3$ is s.s and iso to $M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C})$.

Next time: We'll start proving these facts!