AA2. Lecture 23. 03.11.2022.

Last time: Midterm.

Today:

- proof of the Jordan-Hölder Theorem

Thm: Let A be a k-alg and V an A-module of finite length.

Let $\quad 0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_n = V \qquad (I)$

and $\quad 0 = W_0 \subset W_1 \subset W_2 \cdots \subset W_m = V \qquad (II)$

be two comp. series of V. Then (I) and (II) are equivalent.

- properties of module lengths

1. Pf of the Jordan-Hölder Thm

$$0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_n = V \quad (\mathrm{I})$$
$$0 = W_0 \subset W_1 \subset W_2 \cdots \subset W_m = V \quad (\mathrm{II})$$

Pf: We prove the theorem by induction on $\max(n,m)$, which we assume to be n.

i.e., WLOG, we assume $m \le n$.

Base cases: $n = 0 \Rightarrow V = 0$ and we are done.

$n = 1 \Rightarrow V$ is simple $\Leftrightarrow m = 1$. Done.

Inductive Step: Now assume $n > 1$. We treat two subcases.

Subcase 1 (easy): $V_{n-1} = W_{m-1}$

$$\underline{0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1}} \subset V_n = V$$
$$\underline{0 = W_0 \subset W_1 \subset \cdots \subset W_{m-1}} \subset W_m = V.$$

Since $\max(n-1, m-1) < \max(n,m)$, the underlined comp series of $V_{n-1} = W_{m-1}$ are equivalent by induction, so it follows that (I) and (II) are equivalent.

Subcase 2: $V_{n-1} \neq W_{m-1}$. Then $V_{n-1} + W_{m-1} \supsetneq V_{n-1}$ and hence $V_{n-1} + W_{m-1} = V$.

Let $D = V_{n-1} \cap W_{m-1}$. Then by the 2nd Iso Thm,

$$V/V_{n-1} = V_{n-1} + W_{m-1}/V_{n-1} \cong W_{m-1}/D \qquad (i)$$

and $$V/W_{m-1} = V_{n-1} + W_{m-1}/W_{m-1} \cong V_{n-1}/D \qquad (ii).$$

2nd Iso Thm.

$$\frac{V_{n-1}+W_{m-1}}{V_{n-1}} = W_{m-1}/D$$

for $D = V_{n-1} \cap W_{m-1}$

Now take a comp series $0 = D_0 \subset D_1 \subset \cdots \subset D_t = D$ of D with $t < n$, which exists by the inheritance lemma. By (i) and (ii), the above series induces two comp. series

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = D \overset{(ii)}{\subset} V_{n-1} \subset V \qquad (\mathrm{III})$$

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = D \overset{(i)}{\subset} W_{m-1} \subset V \qquad (\mathrm{IV})$$

of V. Now look at

$$0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_{n-1} \subset V \qquad (\mathrm{I})$$

$$0 = W_0 \subset W_1 \subset W_2 \cdots \subset W_{m-1} \subset V \qquad (\mathrm{II}).$$

Strategy:

(I) ~ (III) and (II) ~ (IV) by induction,

(III) ~ (IV) by construction and (i), (ii),

So (I) ~ (II).

Consider the comp. series of V_{n-1} obtained from (I) and (III) by the obvious truncations and the comp series of W_{m-1} obtained from (II) and (IV) similarly.

We can now apply induction to conclude $\underbrace{n-1 = t+1 = m-1}_{\Downarrow \atop m=n,\ t=n-2=m-2.}$ and that we have equalities of multisets

$$\left[V_{n-1}/D,\ D_i/D_{i-1}\ (1\le i\le \underset{n-2}{t})\right] = \left[V_i/V_{i-1}\ (1\le i\le n-1)\right]$$

and $\left[W_{m-1}/D,\ D_i/D_{i-1}\ (1\le i\le \underset{n-2}{t})\right] = \left[W_i/W_{i-1}\ (1\le i\le m-1)\right]$,

i.e., the composition factors in (I) are

$$V/V_{n-1},\quad V_{n-1}/D,\quad D_t/D_{t-1},\ \ldots\ D_2/D,$$

...... (i) (ii) --- (II) are ...

$$V/W_{m-1},\quad W_{m-1}/D,\quad D_t/D_{t-1},\ \ldots\ D_2/D.$$

The green lines connect iso. modules, so (I) and (II) are equivalent. □

2. Properties of module lengths

Prop: (Prop 3.17. "length behaves well") Let V be an A-module of finite length.

Then for every submodule $U \leq V$ we have:

(1) V/U has a comp. series. (So quotients of fin-len. modules have fin. length.)

(2) There exists a comp. series of V with U as one of its terms.

Moreover, we have $\ell(V) = \ell(U) + \ell(V/U)$.

(3) We have $\ell(U) \leq \ell(V)$, where equality holds iff $U = V$.

Pf: Note that (3) follows immediately from (2): $\ell(V) = \ell(U) + \ell(V/U) \geq \ell(U) + 0 = \ell(U)$

and equality holds iff $\ell(V/U) = 0$, iff $V/U = 0$, iff $V = U$.

(1). Take a composition series $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of V. Consider the

chain $0 = U/U = V_0+U/U \subset V_1+U/U \subset V_2+U/U \subset \cdots \subset V_{n-1}+U/U \subset V/U$. (*)

WLOG, we may assume duplicate terms have been removed from *).

Now, under this assumption we must have $\left(V_i+U/U\right) / \left(V_{i-1}+U/U\right) \cong V_i+U/V_{i-1}+U \underset{\text{Ex.}}{\cong} V_i/V_{i-1}$

for all i, so *) is a comp. series.

(2). (Sketch). By the inheritance lemma, U has a comp series $0 = 0 = U_0 \subset U_1 \subset \cdots \subset U_k = U$.

We hope to extend this series to one for V of the form

$$0 = U_0 \subset U_1 \subset \cdots \subset U_k = U \subset \overset{?}{\cdots} \subset V.$$

Take a comp series of V/U, which exists by (1) and has the form

$0 = U/U \subset W_1/U \subset \cdots \subset V/U$. We can fill "?" by lifting the W_i's there by the correspondence theorem. $\square$