AA2. Lecture 23.

HW deadline moved to this Saturday.

03.09.2022.

Last time:

- Ex. 3.11.(a)
- Existence: Any fin. dim A-module has finite length (for a k-algebra A).
- Inheritance: If an A-module has a comp. series, then so does every submodule of it.

Todo:

- proof of the Jordan-Hölder Thm (uniqueness)

Today:

- HW/Midterm questions

Q: Equivalence between A-modules and representations of A. (Thm 2.33)

A module:

vector space V with a <u>nice</u> A-action

<u>nice:</u> $\boxed{(1)}$ $\forall a \in A$, a acts as a linear map on V. $a\cdot(cv+u) = ca\cdot v + a\cdot u$

$\boxed{(2)}$ · 1_A acts as id. $1_A \cdot v = v \quad \forall v \in V$

· the action assignments respect $+$ and $\cdot$:

$$(a+b)\cdot(v) = a\cdot v + b\cdot v$$
$$(ab)\cdot v = a\cdot(b\cdot v) \qquad \forall a,b \in A,\ v \in V$$

Representations of A:

vect. space V w/ an <u>alg. hom</u> $\rho: A \to \mathrm{End}(V)$

$\rho(1) = \mathrm{id}$.

$\rho(a)+\rho(b) = \rho(a+b)$

$\rho(ab) = \rho(a)\rho(b)$

module $V \longmapsto V$, $\rho: A \to \mathrm{End}(V)$, $a \mapsto$ the action of a, i.e., <u>the map $V \to V$</u>, $v \mapsto a\cdot v \ \forall v \in V$.

it's an algebra hom by (2)

$\in \mathrm{End}(V)$ by (1)

V, $A \curvearrowright V$: $a\cdot v = \rho(a)(v) \ \forall a \in A, v \in V \longleftarrow$ rep V, $\rho: A \to \mathrm{End}(V)$

satisfies (1), (2) since ρ is a hom into $\mathrm{End}(V)$

E.g. $kS_3 \circlearrowright V = k^3 = \text{Span}\langle e_1, e_2, e_3 \rangle$

to describe the module

$V = k^3$.

suffices to describe the action of basis elts of $\underline{A = kS_3}$ on $\underline{\text{basis elts of } V = k^3}$

$\{g : g \in S_3\}$ $\qquad \{e_1, e_2, e_3\}$

$g \cdot \underline{\underline{e_i}} = e_{\underline{\underline{g(i)}}}$

$\longrightarrow$ the corresponding rep

$V = k^3$.

need to describe alg. hom

$\rho : A \to \text{End}(V) = M_3(k)$

w.r.t. $\{e_1, e_2, e_3\}$

$$g \mapsto \rho(g) = \begin{pmatrix} \text{the map sending} \\ \text{sending } e_i \text{ to} \\ e_{g(i)} \ \forall i \end{pmatrix}$$

e.g. $= \begin{pmatrix} \text{the lin. map} \\ e_1 \mapsto e_3 \\ e_3 \mapsto e_2 \\ e_2 \mapsto e_1 \end{pmatrix}$ if $g = (132)$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Q: Ex 3.11.(b). $A = M_2(k)$ ε : idempotent not equal to 0 or 1 in A.

$$\varepsilon_c = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} \qquad \varepsilon_c^2 = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}$$

(a) $(*)$ $0 \subset A\varepsilon \subset A$ is a comp series of $A \curvearrowright A$.

Strategy: · Note that $0 \subset C_1 \subset A$ where $C_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a, b \in k \right\}$ is a comp. series of A, hence $\text{length}(A) = 2$.

· check that $A\varepsilon$ is a nonzero proper submodule of A, so the containments in $(*)$ are proper.

· Deduce that $(*)$ is a comp. series.

(b) $A\varepsilon_\lambda \neq A\varepsilon_\mu$ if $\lambda \neq \mu$. → This is only claim about set (in)equality.

$$\left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \cdot \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix} : x, y, z, w \in k \right\} = \left\{ \begin{bmatrix} x & \lambda x \\ z & \lambda z \end{bmatrix} : x, y \in k \right\} = \cdots$$

Q: Strategies for showing a submodule $\mathcal{U}$ is maximal in a module V:

- show $V/\mathcal{U}$ is simple, e.g. by noting $\dim V - \dim \mathcal{U} = 1$ (if that's the case)

Common

- show that there is no submodule W of V s.t. $\mathcal{U} \subsetneq W \subsetneq V$, by showing that any submodule containing $\mathcal{U}$ and an elt $w \in V \setminus \mathcal{U}$ must be all of V. (e.g. "$J_i \subseteq Ae_i^{\geq 1}$")

- others, like how we showed $A\varepsilon$ is max. in A on the last page.