AA2. Lecture 22.

03.07.2022.

Last time:

- Def of composition series and lengths of modules
- Main results: existence and uniqueness of comp. series of finite dimensional modules
- Examples: zero modules have length 0, simple modules have length 1, k-modules, i.e., k-vec space, have length equal to their k-dim.

Today:

- HW 7. 4(b)
- proofs of the main results.

0. HW7.4(b)

$x := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Key: Show that the $k[x]$-module $k[x] \circlearrowright V = k^2$ is indecomposable but not simple.

Hint: "Not simple": show that $U := \mathrm{Span}\langle e_1 \rangle$ is a submodule.

"indecomposable": by def, it suffices to show that we cannot have two submodule M, N that are proper and nonzero s.t. $V = M \oplus N$.

It in turn suffices to show that every nonzero submodule of V must contain the vector e_1.

why?

why?

1. Existence (easy) Let A be any k-algebra.

Lemma: (Lemma 3.9) Every fin. dim. A-module V has a comp. series (and hence finite length).

Note: The fin. dim. assumption is necessary: if $A = k$ and V an inf dim. k-vec. space then V has no comp series, because the existence of a comp series $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ implies that $\dim(\underbrace{V_i/V_{i-1}}_{\text{simple v.s.}}) = 1$ and hence $\dim V = n < \infty$.
$\quad\quad \| \; \dim V_i - \dim V_{i-1}$

Pf: We use induction on $\dim V =: d$.

Base cases: $d=0$ or $d=1 \Rightarrow V = 0$ or V is simple $\Rightarrow V$ has a comp series as discussed last time.

Inductive step: Say $d > 1$ (and assume any v.s. of dim smaller than d has a comp series). If V is simple, then we are again done since $0 \subset V$ is a comp series of V. So assume V is not simple. Take a maximal proper submodule, U, of V. Then $\dim U < \dim V = d$, so by induction U has a comp $0 = U_0 \subset U_1 \subset \cdots \subset U_m = U$. Since V/U is simple (as U is max. in V), the series $0 = U_0 \subset \cdots \subset U \subset V$ is a comp series for V. $\square$

2. Uniqueness

A : k-algebra

Thm: (Jordan-Hölder Thm, Thm 3.11) Suppose an A-module V has two comp series

$$0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_{n-1} \subset V_n = V \qquad (\mathrm{I})$$

$$0 = W_0 \subset W_1 \subset W_2 \cdots \subset W_{m-1} \subset W_m = V \qquad (\mathrm{II})$$

Then the two series are equivalent, i.e., $n = m$ and there is a permutation $\sigma \in S_n = S_m$ s.t.

$$V_i / V_{i-1} \cong W_{\sigma(i)} / W_{\sigma(i)-1} \quad \forall\, 1 \le i \le n.$$

Rmk: The pf will use induction and the 2nd Iso Thm for A-module:

Let U, W be submodules of an A-module V, then

$$\begin{tikzcd}[column sep=small] & U+W \arrow[dl, dash] \arrow[dr, dash] & \\ U \arrow[dr, dash] & & W \arrow[dl, dash] \\ & U \cap W & \end{tikzcd}$$

$$(U+W)/W \cong U/U\cap W \quad \text{as } A\text{-modules.}$$

Lemma: (inheritance, Prop 3.10) If an A-module V has a comp series, then any submodule $U \subseteq V$ has a comp series.

$\boxed{\text{Pf of the theorem: next time}}$

Pf: Take a comp. series $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of V.

Intersecting each term with U yields a chain of subspaces

$$(*) \quad 0 = (V_0 \cap U) \subset (V_1 \cap U) \subset \cdots \subset (V_n \cap U) = (V \cap U) = U.$$

Removing duplicate terms if necessary, we may assume that all the above containments are proper. We claim that $V_i \cap U / V_{i-1} \cap U \cong \underbrace{V_i / V_{i-1}}_{\text{simple}}$, so $(*)$ is a comp series of U.

Pf of the claim:
$$V_i \cap U / \underline{V_{i-1} \cap U} = (V_i \cap U) / \underline{V_{i-1} \cap (V_i \cap U)} \overset{\text{2nd Iso Thm}}{\cong} V_{i-1} + (V_i \cap U) / V_{i-1} \subset V_i / V_{i-1}.$$

Now, $\left\{ \begin{array}{l} V_{i-1} \cap U \subsetneq V_i \cap U \Rightarrow V_i \cap U / V_{i-1} \cap U \neq 0 \\ V_i / V_i \text{ is simple} \end{array} \right\}$, so $V_i \cap U / V_{i-1} \cap U$ must be iso to V_i / V_{i-1}.