AA2 - Lecture 21.

03.04.2022.

Last time:

- Classification of simple modules of $k[x]/\langle f\rangle$

- Construction of simple modules of kQ for an acyclic quiver Q.

  - proved: $\forall i \in Q_0$, $S_i = Ae_i / \underbrace{Ae_i^{\geq 1}}_{J_i} = \mathrm{Span}\langle e_i + J_i\rangle$ is a simple module of kQ, and $S_i \not\cong S_j$ if $i \neq j$.

  - to prove: **Thm 1**: every simple of kQ is iso. to some S_i, $i \in Q_0$.

    prep: (a) $\forall i \in Q_0$, $e_i A e_i = \mathrm{Span}\langle e_i\rangle$ and $\dim(e_i A e_i) = 1$

    (b) J_i is the only max. submodule of Ae_i, and $e_i J_i = 0$.

Today: · finishing the proof of Thm 1 · Composition series and length of modules

1. Pf of Thm 1. (Q: acyclic quiver. $A = kQ$)

Pf: Let S be a simple module of kQ. We want to show that $S \cong S_i$ for some $i \in Q_0$. To do so, take a nonzero elt $s \in S$.

Then $0 \neq s = 1_{kQ} \cdot s = \left(\sum_{i \in Q_0} e_i\right) \cdot s = \sum_{i \in Q_0} (e_i \cdot s)$,

so for some i we must have $e_i \cdot s \neq 0$. Pick such an i.

Then by Lemma 3.3, we have $S = A(e_i \cdot s) = (Ae_i) \cdot s$ since S is simple.

Now consider the map $\varphi : Ae_i \to Ae_i \cdot s = S$, $ae_i \mapsto ae_i \cdot s$.

It can be easily checked that φ is an A-module hom, so by the 1st Iso Thm we have $Ae_i / \ker\varphi \cong \operatorname{Im}\varphi = Ae_i \cdot s = S$.

since φ is clearly surj by its def

Since S is simple, $Ae_i/\ker\varphi$ is simple, so $\ker\varphi$ is a max. submodule of Ae_i.

By (b), we must have $\ker\varphi = J_i$ and hence $S \cong Ae_i / J_i = S_i$, so we are done. $\square$

2. Composition series and length

Definitions: Let A be a k-alg and V an A-module.

- A <u>composition series</u> of V is a finite chain $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ of submodules of V s.th $\underbrace{V_i/V_{i-1} \text{ is simple}}_{\Updownarrow}$ for all $0 < i \leq n$. The <u>length</u> of such a composition series is n,
  V_{i-1} is a maximal submodule of V_i
  the number of quotients
- We say V <u>has finite length</u> if it admits a composition series.
- If $0 \subset V_0 \subset V_1 \cdots \subset V_n = V$ is a comp. series of V, we call $V_0, \dots, V_n$ the <u>terms</u> of the series and the quotients V_i/V_{i-1} $(0 < i \leq n)$ the <u>composition factors</u> of the series

- We say two composition series $0 = V_0 \subset V_1 \cdots \subset V_n = V$ and $0 = W_0 \subset \cdots \subset W_m = V$ are equivalent if their composition factors are the same up to iso. and reordering, i.e., if the multisets $[V_i/V_{i-1} : 0 < i \leq n]$ and $[W_i/W_{i-1} : 0 < i \leq n]$ are equal.

  Note: two equivalent comp. series must have the same length.

Main results: (proofs to come)

(1) Existence (Lemma 3.9) Every finite dimensional A-module has a comp. series.
(so it has finite length)

(2) Uniqueness (Thm 3.11, Jordan-Hölder Thm) Suppose an A-module V has (at least) a composition series. Then every two comp. series of V must be equivalent.

Def: Suppose V has finite length. Then we define the length of V to be the shared length of all composition series' of V.

Examples: · A: an arbitrary algebra. $V = 0$ then $0 = V_0 = V$ is the unique comp. series of V, so V has finite length and $\mathrm{length}(V) = 0$.

· A: an arbitrary algebra. V: simple A-module.

Then $0 = V_0 \subset V_1 = V$ is a comp. series of V since $V_1/V_0 = V/0 \cong V$ is simple, so simple modules have finite length and length 1.

Note: In general, composition series detect how modules are "built-up" from simple modules, and length of modules detects "distance from being simple".

· Consider the natural module $M_n(k) \curvearrowright \underbrace{k^n}_{V}$, which we showed is simple.

Thus $\mathrm{length}_A(V) = 1$. But $\dim_k(k^n) = n$, so the length of an A-module may be different from its dim. as a k-vector space.

- $\underline{A = k}$ $\quad V = k^n$, $\quad n \in \mathbb{Z}_{>0}$.

(so A-modules are just k-vect. spaces, which are simple iff they are of dim 1)

Then for any basis $\{b_1, \dots, b_n\}$ of V, if we let $V_i = \mathrm{Span}\{b_j : 0 < j \le i\}$.

then $\quad 0 = V_0 \subset \underset{\langle b_1 \rangle}{V_1} \subset \underset{\langle b_1, b_2 \rangle}{V_2} \subset \cdots \subset V_n = V$ is a comp. series of V

because $\dim\left(V_i / V_{i-1}\right) = \dim V_i - \dim V_{i-1} = 1 \quad \forall\, 0 < i \le n$.

So finite dim. v.s have finite length as k-modules, and their lengths as k-modules agree with their dimensions.

- $A = k \times k$; $V = A$, the regular module $(A \circlearrowleft A = \{(x,y) : x \cdot y \in A\})$

Let $S_1 = \{(x,0) : x \in k\} \subseteq A$, $S_2 = \{(0,y) : y \in A\} \subseteq A$.

Ex: · S_1 and S_2 are both submodules of A. Moreover, they are simple.

· $V/S_1 \cong S_2$, $V/S_2 \cong S_1$ as modules

· $S_1 \not\cong S_2$ as A-modules .

Point: It follows that $0 = V_0 \subset V_1 = S_1 \subset V_2 = V$ and

$$0 = W_0 \subset W_1 = S_2 \subset W_2 = V$$

give us two distinct but equivalent comp series with comp factors $\{S_1, S_2\}$.

Next time: · more examples, proofs