AA2. Lecture 2. Discord: https://discord.gg/EMqmNtKw 01. 12. 2022.

Last time:

- Def. of gps and gp homs.
- Working with axioms: e.g. proved uniqueness of inverses in gps.

  E.x. In the def of a gp hom from Lecture 1, Condition (1) is implied by Condition (2) and is hence redundant.

Today. more review:

- gp isomorphism theorems
- rings and fields
- vector spaces and linear algebra

Common theme:

- axiomatic approach
- "categorical notions" (e.g. homs)

1. Groups

We've defined gps and gp homomorphisms.

Now we recall the four gp isomorphism theorem.

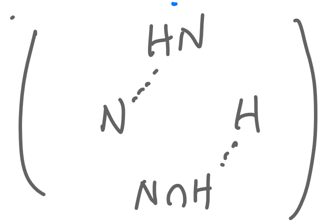
Theorem (1) Let $\varphi: G \to H$ be a gp hom. Then there is a well-defined gp isomorphism $\bar{\varphi}: G/\ker\varphi \to \operatorname{Im}\varphi$ given by the formula

$$\bar{\varphi}(g \cdot \ker\varphi) = \varphi(g) \qquad \forall g \in G$$

(2) Let G be a gp, H a subgp of G, N a normal subgp of G.

Then $HN/N \cong H/N \cap H$ as gps.

$HN := \{hn : h \in H, n \in N\}$



Ex: Prove (2) by using (1). find an iso.

13) Let G be a gp, H and N normal subgps of G s.t. $N \subseteq H$.
(i.e., N is a subset of H, which is equivalent to $N \leq H$ (subgp) since N is a subgp of G.)

Then $\dfrac{G/N}{H/N} \cong G/H$.

14) (a.k.a. the Correspondence Thm) Let G be a gp and N a normal subgp of G. ("$N \trianglelefteq G$")

Then there is a bijection

$$\left\{ \begin{array}{l} \text{subgps of } H \\ \text{s.t. } N \subseteq H \leq G \end{array} \right\} \longrightarrow \left\{ \text{subgps of } G/N \right\}$$

$$H \longmapsto H/N$$

Ex: Describe/find the inverse of this bijection.

2. Rings

Def: A ring is a triple $(R, +, \cdot)$ where

- $(R, +)$ is an abelian gp. → expands to five axioms. (in particular, R has a $+$ identity, 0)
- $\cdot$ is a binary operation on R.
- $\cdot$ is associative, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in R$.
- $\cdot$ distributes over $+$, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a+b) \cdot c = a \cdot c + b \cdot c$ $\forall a, b, c \in R$.
- R contains an elt, 1, s.t. $1 \neq 0$ and $a \cdot 1 = a = 1 \cdot a$ $\forall a \in R$.

→ the "unit"

Remarks:

(a) The last axiom is sometimes not required for rings, but we will always require it, i.e., we'll assume our rings always have a unit, in this course.

(b) A ring R ($(R,+,\cdot)$) is called <u>commutative</u> if $a\cdot b = b\cdot a$ $\forall a,b\in R$.

(c). Ex: Show that if "$1=0$", then $R=\{0\}$. (HW)

Examples: $(\mathbb{Z},+,\cdot)$, $(\mathbb{Q},+,\cdot)$, $(M_n(\mathbb{R}),+,\cdot)$ form rings.

$M_n(\mathbb{R})$: $n\times n$ matrices over $\mathbb{R}$

Ex: $M_n(\mathbb{R})$ is commutative $\iff$ $n=1$.

Def: (ring hom) A map $f: R \to S$ between two rings R and S is a ring homomorphism if

(1) $f(0_R) = 0_S$ (2) $f(a+b) = f(a) + f(b)$

(3) $f(1_R) = 1_S$ (4) $f(ab) = f(a) f(b)$

for all $a, b \in R$.

Rmk: There are isomorphism / correspondence theorems for rings as wells.

Review their statements.

3. Fields.

Ex: Expand this to a list of axioms.

Def: A field is a (unital) commutative ring $(F, +, \cdot)$ where every elt $a \in F \setminus \{0\}$ has a multiplicative inverse b, i.e. an elt b s.t.

$$a \cdot b = 1 = b \cdot a.$$

Examples and nonexamples:

$(\mathbb{Z}, +, \cdot)$?

$(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$?

$M_n(\mathbb{R})$, $n \geq 2$?

$GL_n(\mathbb{R})$, $n \geq 2$?

Next time:

- vector spaces and linear algebra