AA2. Lecture 19.

Midterm (take-home): available on Canvas from 5 pm, Mar. 9 to 5 pm, Mar. 10.

02. 28. 2022.

Last time:

- Lemma 3.3: the "(strong) cyclicity" test for simplicity
  An A-module V is simple $\iff$ $\forall w \in V \setminus \{0\}$, $Aw = V$.

- Examples and nonexamples of simple modules

Today:

- simplicity of quotient modules

- simple modules of "$k[x]/\langle f \rangle$" (§ 3.4.1)

1. Simplicity of quotient modules

Let A be a k-alg.

Q: Let V be an A-module and $U \subseteq V$ a proper submodule.

When is V/U simple, i.e., when are V/U and $U/U = 0$ the only submodules of V/U?

Recall that there is an order-preserving bijection

$$\{\text{submodules of } V/U\} \longleftrightarrow \{\text{submodules of } V \text{ containing } U\}$$

$$U/U \subseteq W/U \subseteq V/U \longleftarrow U \subseteq W \subseteq V$$

The following is now immediate:

Prop: The module V/U is simple iff U is a maximal submodule of V, i.e., iff the only submodules W w/ $U \subseteq W \subseteq V$ are U and V.

A related useful fact. all simple modules of A (k-algebra) are of the form up to iso, A/M where M is a maximal submodule of A (regarded as a regular module), i.e., of the form A/M where is a maximal ideal of A.

Prop 1: Let V be a simple A-module. Then there is a module isomorphism

$$V \cong A/M$$

where M is a maximal ideal of A.

Pf: Take any $w \in V \setminus \{0\}$. Then we have an surj. A-module hom

$$\varphi: A \to Aw, \quad a \mapsto a \cdot w$$

By Lemma 3.3, we have $Aw = V$, so this is a hom $\varphi: A \to V$. and we have

$V = Aw = \operatorname{Im}\varphi \cong A/\ker\varphi$ (where $\ker\varphi = \{a \in A : a \cdot w = 0\} = \operatorname{Ann}(w)$).

Since V is simple, $A/\ker\varphi$ is simple, so $\ker\varphi$ is maximal. $\to$ Take $M = \ker\varphi$. $\square$

2. Simple modules of "$k[x]/\langle f\rangle$"

Recall that a polynomial $f \in k[x]$ is called irreducible if for any factorization $f = gh$, then g or h is a constant. (e.g. x^2+1 is irr. in $\mathbb{R}[x]$, but reducible in $\mathbb{C}[x]$ since $x^2+1 = (x-i)(x+i)$.)

Also note/recall that

(*) For two ideals $\langle f\rangle, \langle g\rangle \subseteq k[x]$, we have $\langle f\rangle \subseteq \langle g\rangle$ iff $g \mid h$.

If $g \mid f$, $k[x]/\langle g\rangle$ is naturally a $k[x]/\langle f\rangle$-module under the action

$$(p+\langle f\rangle)(q+\langle g\rangle) \overset{\circledast}{=} pq + \langle g\rangle.$$

because (1) $\underbrace{k[x]/\langle g\rangle}_{M}$ is a $\underbrace{\text{natural } k[x]\text{-module}}_{\text{it's a quotient of } k[x] \curvearrowleft k[x]}$ under $p(q+\langle g\rangle) = pq+\langle g\rangle$

(2) Everything in the ideal $I := \langle f\rangle$ acts as 0 on $\underbrace{k[x]/\langle g\rangle}_{M}$. So the action in (1) induces a $k[x]/\langle f\rangle$ on M.

We are ready to prove ......

Prop ($\approx$ Prop 3.23). Let $A = k[x]/\langle f \rangle$ with $\langle f \rangle$ of positive degree.

1) Up to isomorphism, the simple A-modules are precisely the A-modules $k[x]/\langle h \rangle$ where h is an irreducible polynomial dividing f.

pf: • If h is irreducible, then $\langle h \rangle$ is a maximal ideal in $k[x]$. It follows that $k[x]/\langle h \rangle$ is a simple $k[x]$-module, and hence a simple $k[x]/\langle f \rangle$-module (under $\circledast$).

EX, use Lemma 3.3

• Let S be a simple A-module. Then by Prop I, $S \cong A/U$ for a maximal submodule U of A. Since U is a maximal submodule of $A = k[x]/\langle f \rangle$, it must be of the form $\langle h \rangle/\langle f \rangle$ for some irreducible polynomial h dividing f. Thus, we have $S \cong A/U = \dfrac{k[x]/\langle f \rangle}{\langle h \rangle/\langle f \rangle}$. By the 3rd iso thm, the last quotient is iso to $k[x]/\langle h \rangle$ via the $k[x]$-module iso $\bar{\varphi}$ induced by the map $\varphi : k[x]/\langle f \rangle \to k[x]/\langle h \rangle$, $g + \langle f \rangle \mapsto g + \langle h \rangle$.
φ is also an A-module hom, so $S \cong \dfrac{k[x]/\langle f \rangle}{\langle h \rangle/\langle f \rangle}$ as A-modules. $\square$

Example: $k = \mathbb{C}$. $\quad A = k[x]/(x^2+1) \quad x^2+1 = (x-i)(x+i)$

By the Prop, the simples of A are precisely $k[x]/\langle x-i \rangle$ and $k[x]/\langle x+i \rangle$, where A acts by $\star$. But why are these two simples non-isomorphic?

↓

Next time: Part (2) of the prop.

Simples of path algebras