AA2. Lecture 18.

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Last time:

- gp actions $G \curvearrowright X$ gives rise to gp rep $G \curvearrowright kX$.
- def. of simple modules
  - e.g. · For an arbitrary algebra A, modules of dim. 1 are always simple.
    - For $A = k$, an A-module is just k-vector space V, and V is simple iff $\dim V = 1$, i.e., if $V = k$.
    - $M_n(k) \curvearrowright k^n$ is simple.
- a strategy for proving a module V is simple: prove that every nonzero elt in V generates all of V.

Today:

- Lemma 3.3.
- More examples, including simple modules of quotient modules.

1. Lemma 3.3.

Lemma ("cyclicity test for simplicity") Let A be an $\Bbbk$ algebra and V a nonzero A-module. Then V is simple if and only if $\forall w \in V \setminus \{0\}$ we have $Aw = V$.
$\Leftarrow$ $\Rightarrow$

Pf: ($\Rightarrow$). Suppose V is simple and take $w \in V \setminus \{0\}$. Then Aw is a submodule of V, so $Aw = 0$ or $Aw = V$. But $0 \neq w = 1 \cdot w \in Aw$, so $Aw \neq 0$. Therefore $Aw = V$.

($\Leftarrow$) Suppose $Aw = V \quad \forall w \in V \setminus \{0\}$. Let U be a nonzero submodule of V. We want to show that $U = V$. Take a nonzero elt $w \in U$. By supposition, we have $V = Aw \subseteq U$, so $U = V$. as desired. $\square$

2. More examples

$\begin{bmatrix}1\\1\\0\end{bmatrix}$

(a) (Regular modules of division algebras) Let D be a division algebra (so that every nonzero elt in D has a mult. inverse). Then the regular module $D \curvearrowright D$ is simple: Let $u \in D \setminus \{0\}$. Then u has an inverse so $1 = u^{-1} \cdot u \in Du$.

It follows that $\underset{d}{D} = \underset{d \cdot 1}{D \cdot 1} \subseteq D(Du) \subseteq Du$, therefore $D = Du$. It follows from Lemma 3.3 that D is simple.

We have actually shown that $\mathrm{Span}(e_1+e_2+e_3)$ is a submodule of k^3.

(b) Is the natural perm. module k^n of S_n simple for $n \geq 2$?

e.g. $n=3$. $kS_3 \curvearrowright k^3$ $\quad (23) \cdot \begin{bmatrix}4\\1\\7\end{bmatrix} = (23)(4e_1 + e_2 + 7e_3) = 4e_1 + e_3 + 7e_2 = \begin{bmatrix}4\\7\\1\end{bmatrix}$

Lemma 3.3: This is equivalent to asking if every nonzero elt in k^3 generates k^3.

Answer: No: Note that $\pi(e_1+e_2+e_3) = e_{\pi(1)} + e_{\pi(2)} + e_{\pi(3)} = e_1 + e_2 + e_3 \ \forall \pi \in S_n$, so

$\left(\sum_{\pi \in S_n} \lambda_\pi \pi\right) \cdot (e_1+e_2+e_3) = \lambda(e_1+e_2+e_3)$ for some scalar λ. So $(kS_3) \cdot (e_1+e_2+e_3) = \mathrm{Span}(e_1+e_2+e_3) \neq k^3$.

(c) (Ex 3.6. / HW) $\left(A = kQ \text{ for } Q = \overset{x}{\circlearrowleft} \bullet \circlearrowright^{y} \right) \circlearrowright k^n$.

Fact: For any choice of $X, Y \in M_n(k)$, taking x, y to X, Y respectively extends to an algebra hom $\theta : A = kQ \to \mathrm{End}(k^n)$,

$x \mapsto (\text{mult. by})\ X$, $y \mapsto Y$.

e.g. $2xy - y^2x \mapsto 2XY - Y^2X$.

hence we get an A-module structure on k^n.

$x = (X \cdot -)$, $y = (Y \cdot -)$

$\circlearrowright V = k^n$ → denote by $V_{X,Y}$

The exercise: show that when $n = 3$, $V_{X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} = \ X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \circlearrowright k^3 \circlearrowleft Y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is simple.

Q: Take $u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in k^3 \setminus \{0\}$. How do we hit it with A and get every vec. in k^3?

X: $e_1 \mapsto 0$, $e_2 \mapsto e_1$, $e_3 \mapsto e_2$ "raising"

Y: $e_1 \mapsto e_2$, $e_2 \mapsto e_3$, $e_3 \mapsto 0$. "lowering"

$\Rightarrow$ "If we have e_1 or e_3, we can get everything"

Example calculation:

$$\begin{bmatrix}4\\1\\7\end{bmatrix} \xrightarrow{X,\ \text{raise}} \begin{bmatrix}1\\7\\0\end{bmatrix} \xrightarrow{X,\ \text{raise}} \begin{bmatrix}7\\0\\0\end{bmatrix} \xrightarrow{\frac{1}{7}\cdot 1_A} \begin{bmatrix}1\\0\\0\end{bmatrix} = e_1 \checkmark$$

$= e_1$, essentially

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{Y,\ \text{lower}} \begin{bmatrix}0\\1\\0\end{bmatrix} \checkmark \xrightarrow{Y,\ \text{lower}} \begin{bmatrix}0\\0\\1\end{bmatrix} \checkmark$$

↓

generalizing this example to arbitrary $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$

will prove that $V_{X,Y}$ is simple.

Ex. 3.6 (b) & (c). Construct modules of arbitrary dim (as allowed) of kQ.

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One way: still let X and Y act as raising and lowering operators

e.g. $n=5$, $V=k^5$. $x = X = \begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$, $Y = \dots$.

Next time: simplicity of quotient modules, simple modules of $k[x]/\langle f\rangle$.