AA2. Lecture 17. 02.23.2022.

Last time:

- preservation of scaling action,
- gp representations, and how they are "the same" as reps of gp algebras.

$$\left\{ \begin{array}{c} \text{reps of a gp } G \\ (V,\rho),\ \rho: G \to GL(V) \\ g \mapsto g. \end{array} \right\} \underset{\text{restrict: } \rho := \theta|_G}{\overset{\text{linearly extend the (linear) actions}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{reps of the gp algebra } kG \\ (V,\theta),\ \theta: kG \to \mathrm{End}_k(V) \end{array} \right\}$$

Today.

- from gp actions to gp reps.
- simple modules (Ch. 3)

1. From gp actions to gp reps

Recall from Math 3140 the def. of gp actions:

Def 1: Let G be a gp and X be a set. A (left) group action of G on X is a map $G \times X \to X$, $(g,x) \mapsto g.x$ s.t.

(a) $e_G . x = x \ \forall x \in X$, (b) $(gh).x = g.(h.x)$

Also recall: Given any set X, we can form the free vector space $V = kX$; the elts of kX are formal linear combinations of elts of X.

(e.g. $G \to kG$; $X = \{1,2,3\} \to kX \cong k\langle e_1, e_2, e_3\rangle = k^3$.)

We can now construct a G-module from any G-action on any set X:

Prop: Let $G \curvearrowright X$ be a G-action on a set X. Then there is a rep. (V, ρ) where $V = kX$ and $\rho : G \to GL(V)$ is given by

$$\underset{\bigoplus_{x \in X} kE_x}{\overset{\shortparallel}{}} \qquad \rho(g) = (\text{the unique linear map sending } E_x \text{ to } E_{g \cdot x}).$$

pf: We just need to show that ρ is a gp hom. i.e., $\rho(gh) = \rho(g)\rho(h)$ $\forall g, h \in G$.

To do so it suffices to show that $\rho(gh)(E_x) = \rho(g)[\rho(h)(E_x)]$.

By def of ρ. LHS $= E_{gh \cdot x}$, RHS $= E_{g \cdot (h \cdot x)}$.

By def of gp action, $(gh) \cdot x = g \cdot (h \cdot x)$, so LHS = RHS, as desired. $\square$

E.g. The natural permutation action $S_n \curvearrowright \{1, 2, \dots, n\}$ yields the natural permutation rep $S_n \curvearrowright k^n$ $(\pi \cdot e_i = e_{\pi(i)})$ for all n.

Theme: Modules / reps are about actions:

to define a module structure on a v.s V / to define a rep afforded by V for an algebra / gp is to specify how elts of the algebra/gp act as lin. maps on V.

2. Simple module (Ch. 3) → will be the "building blocks" of modules.

Def: Let A be an algebra. A <u>simple</u> A-module V is an A module that
(1) $V \neq 0$ and (2) V has no proper, nonzero submodule.

(In other words, the simple modules are precisely those with exactly two submodules)

Eg. (a) (modules of dim 1) For any k-algebra A, an A-module V of dimension $\dim_k(V) = 1$ must be simple by dimension considerations.

The converse is not true, that is, simple modules don't have to have dimension 1.
counter-example

(b) $M_n(k) \curvearrowright k^n$:

<u>Claim:</u> The natural module k^n of $M_n(k)$ is simple for all $n \in \mathbb{Z}_{\geq 1}$.

Common proof strategy: To prove a nonzero A-module V is simple, we show that any nonzero submodule W has to be V itself. To do the latter, we show that any nonzero elt v (W must contain a nonzero elt if $W \neq 0$) generates V in the sense that

$$V \subseteq A \cdot v \ (\subseteq W).$$

Pf of claim: Let $W \subseteq k^n$ be a submodule that's nonzero. Then W contains some nonzero elt $v = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \in k^n$. Since $v \neq 0$, we have $\lambda_i \neq 0$ for some i.

How to get all of k^n from v?

e.g. $n=3$, $v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Or, use that $E_{i2} \cdot v = e_i$ $\forall i$ ...

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\text{''}\ \frac{1}{2} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

It follows that $\forall u \in V$, the matrix M whose ith col is $\left(\frac{1}{\lambda_i} u\right)$ and other columns are all zero satisfies

$$M \cdot v = u \quad \text{by linear algebra}$$

So $V = A \cdot v \subseteq W$, so $W = V$.

It follows that V is simple. $\square$

Next time:

- Lemma 3.3 of [EH] (subsumes our "common strategy")
- more simplicity examples / arguments