AA2. Lecture 16.

02-21. 2022

Last time: Two equivalences:

- Given a v.s. V, a $k[x]$-module structure is equivalent to a linear map $\alpha : V \to V$. $\alpha \longmapsto V_\alpha = V^{\,\mathcal{G}\, x \cdot = \alpha(-)}$

- For a quotient A/I of an algebra A by a two-sided ideal I,

$$\{\text{modules of } A/I\} \equiv \{\text{modules of } A \text{ annihilated by } I\}.$$

Eg. For an ideal $I \subseteq k[x]$, I is necessarily $\langle f \rangle$ for some $f \in k[x]$, and

$$\{\text{modules of } k[x]/I = k[x]/\langle f\rangle\} \equiv \{\text{module } V_\alpha \text{ annihilated by } f\}$$

Eg. Restricting to 1-dim modules $V = k$, $\{\text{modules of } k[x]/\langle f\rangle\} = \{\text{modules } V_\alpha \text{ where } x \text{ acts as scaling by a root of } f\}$

Today:
- more on preservation of scaling actions
- representations of gps vs. gp algebras

1. Preservation of scalar actions

Prop: Let A be a k-algebra, let $a \in A$, and let $\varphi: V \to W$ be an A-module hom. If a acts as scaling by a scalar λ on V, then a does the same on $\operatorname{Im}\varphi$.

Pf: Take $w \in \operatorname{Im}\varphi$. Then $w = \varphi(v)$ for some $v \in V$. Thus,

$$a \cdot w = a \cdot \varphi(v) = \varphi(a \cdot v) = \varphi(\lambda v) = \lambda \varphi(v) = \lambda w. \qquad \square$$

Corollary: In the setting of the prop. if φ is an isomorphism (so that $\operatorname{Im}\varphi = W$) and a acts as scaling by λ on V, then a must act as scaling by λ on W as well.

Contrapositive: If V, W are modules on which $a \in A$ act as different scalars, then $V \not\cong W$ as A-modules.

E.g. $V_1 \not\cong V_2$ for $k[x]/x^2-3x+2$ in the example from the last lecture.

2. Group representations

Def: Let G be a gp. A representation of G is the data (V, ρ) of a v.s. V and a gp homomorphism $\rho: G \to GL(V)$ $\equiv GL_n(k)$ once we fix a basis of V.

$\rho(g)\rho(g') = \rho(gg')$

$GL(V) = \{ f: V \to V \mid f \text{ is lin. and invertible} \}$

Example: **Fact:** For $G = S_3$, there is a representation ρ of G afforded by $V = k^3 = \mathrm{span}\langle e_1, e_2, e_3 \rangle$ (chosen basis)

with $\rho(e_G) = \mathrm{id}_V = \left(\text{the linear map sending } \begin{matrix} e_1 \to e_1 \\ e_2 \to e_2 \\ e_3 \to e_3 \end{matrix}\right) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (the linear map $\in GL(V)$)

$$\rho((12)) = (\text{the linear map with } e_1 \to e_2, e_2 \to e_1, e_3 \to e_3) \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and more generally $\rho(\sigma) \overset{*}{=} (\text{the lin. map with } e_i \mapsto e_{\sigma(i)}\ \forall 1 \le i \le n)$.

e.g. $\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Pf of the fact: We just need to show that the assignment $*$ indeed defines a gp hom. ie, that $\rho(\pi)\rho(\pi') = \rho(\pi\pi')$ $\forall \pi, \pi' \in S_3$.

It suffices to show $[\rho(\pi)\rho(\pi')](e_i) = [\rho(\pi\pi')](e_i)$ $\forall\ 1 \le i \le n$,

ie, $\rho(\pi)\left(e_{\pi'(i)}\right) = e_{(\pi\pi')(i)}$, ie. $e_{\pi(\pi'(i))} = e_{(\pi\pi')(i)}$.

The last equation holds since $\pi(\pi'(i)) = (\pi\pi')(i)$, so we are done.

Note: Similarly, for any $n \in \mathbb{Z}_{\ge 2}$, there is a natural "permutation representation"

$$\rho: S_n \to GL(\underbrace{k^n}_{V}), \quad \pi \mapsto (\text{the linear map sending } e_i \text{ to } e_{\pi(i)}\ \forall\ 1 \le i \le n).$$

Q: How are the reps. of a gp G related to reps of kG? A: They are equivalent (2.41).

Prop: Let G be a gp and K a field.

"(b)" Given a rep. $(V, \theta: kG \to \mathrm{End}(V))$ of kG, the restriction

$\rho: G \to GL(V)$, $g \mapsto \theta(g)$ ($G \subseteq kG$) (One needs to check that $\theta(g)$ is invertible) defines a rep of G.

"(a)": Given a rep $(V, \rho: G \to GL(V))$ of G, we can extend ρ linearly to a rep $(V, \theta: kG \to \mathrm{End}(V))$ of kG given by

$$\theta\left(\sum_{g \in G} \lambda_g \cdot g\right) = \sum_{g \in G} \lambda_g \cdot \rho(g).$$

Upshot: reps of G $\equiv$ reps of kG.

E.g. We've talked about the natural module k^3 of kS_3, and we just talked about the natural (perm.) rep. of kS_3. They are both determined by the formula

$$\pi \cdot e_i = e_{\pi(i)}$$

e.g. For the kS_3 module structure, $[3\cdot(12) - 2\cdot(132)]\cdot e_2 = 3e_1 - 2e_1 = e_1$

pf: Ex. $\left(\begin{array}{l} \text{(b): show } \theta(g) \in GL(V), \text{ show } \rho \text{ resp. mult. in } G. \\ \text{(a): show } \theta \text{ respects mult. in } kG \text{ and } \theta(1) = id_V. \end{array}\right)$

3. A word on Ex. 2.21.

$Q: \; 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$

Part (c). Ignore the first part, about "representation of the quiver Q".

but do decide if $V = kQe_2$ is iso to $W = kQ\beta$

are iso as kQ-modules.

Hint: Think about the map $V \to W$, $a \cdot e_2 \mapsto a e_2 \cdot \beta$.

Next time:

- from gp actions to gp reps.
- Simple modules.