AA2. Lecture 15.

02.~~16~~18. 2022.

Last time:

- Recognition theorem for direct sums
- modules of $A = k[x]$

(1) The module action is completely determined by the action of x.

(2) Fix a v.s. V, then the data of an A-module structure on V is equivalent to the data of a linear map $\alpha: V \to V$.

(i) If V is an A-module, then V is determined by the map α corresponding to the action of x by (1).

(ii) Given an <u>arbitrary</u> lin. map $\alpha: V \to V$, there is indeed a well-defined module structure on V where x acts as α. This is defined by $f \cdot v = f(\alpha)(v)$ $\forall f \in k[x], v \in V$.

Pf that "$f \cdot v \overset{*}{=} f(\alpha)(v)$" does make V a $k[x]$-module:

- direct verification: check that the module axioms hold for the action $*$.

- use the representation perspective: to make V a $k[x]$-module is to have an algebra hom $\rho : k[x] \longrightarrow \mathrm{End}_k(V)$. Given an arbitrary $\alpha \in \mathrm{End}_k(V)$, we know (HW) that the evaluation map $\mathrm{Eval}_\alpha : f \mapsto f(\alpha)$ is an alg. hom. from $k[x]$ to $\mathrm{End}_k(V)$. So taking $\rho = \mathrm{Eval}_\alpha$ yields a rep of $k[x]$. In the corresponding module structure on V, every elt $f \in k[x]$ acts as the map $\mathrm{Eval}_\alpha(f) = f(\alpha)$. This is precisely the action given by $*$.

<u>Upshot</u>: Specifying a $k[x]$-module structure on a v.s. V is the same as specifying a linear map on V.

2. Modules of quotients of $k[x]$.

We first note some generalities about modules of a quotient of an alg. A vs. modules of A itself.

Prop 1 Let A be an alg. and I a two-sided ideal of A.

(a) Any module V of A/I is automatically an A-module by the action

$$a \cdot v = (a+I) \cdot v \quad \forall a \in A,\ v \in V$$

(b) Let W be an A-module. Then we $\overline{\text{known}}$ can make W an A/I module via the formula $(a+I) \cdot w = a \cdot w \ \forall a \in A, w \in W$, iff $I \cdot W = 0$.

(i.e., "every elt of I kills all of W")

Upshot: Modules of A/I are the same as those modules of A annihilated by I.

Pf: (1) Again, we can check the module axioms for $a \cdot v \overset{*}{=} (a+I)\cdot v$.

Or, we count note that $*$ simply arises from the representation

$$\begin{array}{ccccc} A & \xrightarrow{\pi_I} & A/I & \xrightarrow[\text{hom since } V \text{ is an } A/I \text{ module}]{} & \mathrm{End}_k(V) \\ a & \longmapsto & a+I & \longmapsto & (a+I)\cdot - \end{array}$$

(2) (Sketch) The key is well-definedness of the proposed action

$$(a+I)\cdot w \overset{\star}{=} a\cdot w$$

$\star$ is well defined $\iff$ whenever $a+I=b+I$ for representatives a,b we have $a\cdot w = b\cdot w$ $\forall w \in W$

$\iff$ whenever $\underbrace{a-b}_{\text{"}i\text{"}} \in I$ we have $\underbrace{(a-b)}_{\text{"}i\text{"}} \cdot w = 0$ $\forall w\in W$

$\iff$ $I \cdot W = 0$.

If / Once $\star$ is well-defined, it's straightforward to check that it satisfies the module axioms, making W an A/I-module.

Back to the case $A = k[x]$: Let I be an ideal of $k[x]$.

Then I must be a principal ideal $I = \langle f \rangle$ for some $f \in k[x]$.

So modules of $k[x]/I = k[x]/\langle f \rangle$ are just modules of $k[x]$ on which f acts as zero. A module of $k[x]$ is necessarily of the form V_α where V is a v.s. and $x \in k[x]$ acts as a linear map $\alpha : V \to V$, and on V_α we know f acts as $f(\alpha)$. The following theorem now follows.

Thm: (Thm 2.10) Let $f \in k[x]$ be non-constant polynomial, and let $B = k[x]/\langle f \rangle$.
(so that $A/\langle f \rangle$ is not trivial)

Then there is a bijection

$$\{ B\text{-modules } V \} \longleftrightarrow \{ k[x]\text{-modules } W = V_\alpha \text{ on which } f \text{ acts as } 0 \}$$

$k[x]/\langle f \rangle$

$V \longmapsto$ use Prop 1.11)

use Prop 1.12) $\longleftarrow W$.

E.g. Say $k = \mathbb{R}$. $A = k[x]$ and $V = k$. Take $f = x^2 - 3x + 2 = (x-1)(x-2)$

Then each linear map on $V = k$ must be scaling $\alpha : k \to k$, $\lambda \mapsto c \cdot \lambda$

for a constant $c \in k$ by linear algebra.

It follows that making V an A-module is just to specify what scalar x should

as on $V = k$. **e.g.** For $c = 3$, V has an A-module structure s.t.

$x \cdot \lambda = 3\lambda$. $\downarrow$ $(x^2 - 3x + 2) \cdot \lambda = (9 - 9 + 2)\lambda$

$V_\alpha = V_{x=3} =: V_3$ $= 2\lambda$.

(**Ex:** If c, c' are distinct constants, then

$x=c \circlearrowright V_c \not\cong V_{c'} \circlearrowleft x = c'$ as module. — see email on HW4.)

By Thm 1, V_c is automatically a $k[x]/\langle f \rangle$-module iff on V_c f acts as 0,

iff $f(c) = c^2 - 3c + 2 = 0$, iff $c = 1$ or $c = 2$.

Conclusion: There are precisely two 1-dimensional $k[x]/\langle f\rangle$ modules up to isomorphism. They are V_c for the roots $c=1$ and $c=2$ of f.

Eg. · $k=\mathbb{R}$. How many 1-dimensional modules of $k[x]/(x^2-1)$ are there up to isomorphism over k?

$x^2-1=(x-1)(x+1)$

Answer: Two: V_1 and V_{-1}

· $k=\mathbb{R}$. 1-dim modules of $k[x]/(x^2+1)$?

They'd be of the form V_c s.t. c is a root of x^2+1 over $k=\mathbb{R}$. There is no such module.

· $k=\mathbb{C}$. 1-dim .... $k[x]/(x^2+1)$? Two: V_i, V_{-i} $(i^2+1=0)$

Next time: gp actions, gp representations, and reps/modules of gp algebras.