AA2. Lecture 14.

02.14.2022.

Last time:

- iso. and correspondence thms for modules
- external vs. internal direct sums

Prop 1. Let M be a module of an algebra A, and $(M_i)_{i\in I}$ submodules of M. Then the map $\varphi: \bigoplus_{i\in I}^{E} M_i \to M$, $(m_i)_{i\in I} \mapsto \sum_{i\in I} m_i$ (sum in M) is an iso, i.e., $M \cong \oplus^{E} M_i$, iff $(M_i)_{i\in I}$ satisfies the conditions in the def. of internal direct sums, i.e., (1) $M = \sum M_i$ (2) $\forall j \in I$, $M_j \cap \sum_{\substack{i\neq j \\ i\in I}} M_i = 0$.

Today:

- more on direct sums
- modules of $k[x]$ and its quotients

1. More on direct sums

Pf: (sketch) Note that

(a) The map φ is a module hom. (EX)

$$\varphi(a\cdot(m_i)_{i\in I}) = \varphi\big((a\cdot m_i)_{i\in I}\big) = \sum_{i\in I} a\cdot m_i = a\cdot\sum_{i\in I} m_i = a\cdot\varphi((m_i)_{i\in I})$$

(b) The map φ is surj. iff $M = \sum_{i\in I} M_i$, i.e., iff Condition (1) holds.

(c) We claim that φ is inj. (equivalently, $\ker\varphi = 0$) iff Condition (2) holds:

"if": Suppose condition (2) holds. To prove $\ker(\varphi)=0$, suppose otherwise and take $0 \neq (m_i)_{i\in I} \in \ker\varphi$. Then for some $j\in I$ we have $m_j \neq 0$.

Since $(m_i)_{i\in I} \in \ker\varphi$, we have $\sum m_i = \varphi((m_i)_{i\in I}) = 0$, so

$m_j = -\sum_{i\neq j} m_i$ where the left side is in M_j and the right side is in $\sum_{\substack{i\in I, \\ i\neq j}} M_i$. By Condition (2), both sides must be 0,

contradicting the assumption that $m_j \neq 0$.

It follows that φ is inj.

"only if": Ex. (similar idea.)

Point of the Prop:

Conditions (1) and (2) are criteria for <u>recognize</u> when M is

isomorphic to $\bigoplus_{i\in I}^{E} M_i$.

Examples of direct sums

(a) Let $A = M_n(k)$ and consider the regular module $A \circlearrowright \underline{\underline{A}}$.

Let $C_i = \left\{ \begin{array}{c} \; i \; \\ \left[\begin{array}{c|c|c} 0 \cdots 0 & * & 0 \cdots \end{array} \right] \end{array} \right\}$ = { matrices whose entries are all zero outside column i } $\forall 1 \le i \le n$.

Recall from linear algebra that $M \cdot \left[v_1 | v_2 | \cdots | v_n \right] = \left[M \cdot v_1 | \cdots | M \cdot v_n \right]$ $\forall M \in A$.

It follows that $M \cdot C_i \subseteq C_i$ and C_i is a submodule of A $\forall i$.

Note that $(C_i)_{1 \le i \le n}$ satisfies conditions (1) & (2), so $A \cong \bigoplus_{i=1}^{n} C_i$ as modules.

Note: Incidentally, all the modules C_i are iso. to the natural module $A \circlearrowright V = k^n$ via the module iso. $\varphi_i : C_i \to V$, $\left[\begin{array}{c|c|c} \underset{0}{\cdots} & v_i & \underset{0}{\cdots} \end{array} \right] \mapsto v_i$

(b) Ex 2.6: The regular module of a path algebra $A = kQ$ for a quiver $Q = (Q_0, Q_1)$ is iso, as a module, to the direct sum $\bigoplus_{i \in Q_0} Ae_i$. (HW)

2. Modules of $k[x]$ and its quotient.

$A := k[x]$

Key observations: The algebra $k[x]$ is generated by 1 and x as an algebra, and on any module M of A 1 has to act as identity, so the action of A on M is completely determined by the action of x.

$(x, v) \mapsto x.v$

Notation: If V is an A-module with x acting as a linear map $(\alpha : V \to V) \in \mathrm{End}_k(V)$, then we denote V by V_α.

E.g. Take $V = k^2$ and let $\alpha : V \to V$ be the map given by $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ w.r.t. to standard basis. **Fact:** We can make V into a module V_α with x acting as α. (Why?)

It follows that in V_α, we'd have

$$\underbrace{(x^2 - 2x)}_{\in A} \cdot \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\in V_\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Prop: Let V be a v.s. and let $\alpha : V \to V$ be an arbitrary linear map.

Then we can make V a $k[x]$-module by declaring that

$$f \cdot v = \underbrace{f(\alpha)}_{\mathrm{Eval}_\alpha(f)} \cdot v \qquad \forall f \in k[x],\ v \in V$$

e.g. $f = x^2 - 2x \longmapsto f$ acts as $f(\alpha) = \mathrm{Eval}_\alpha(f) = \alpha^2 - 2\alpha$.

in particular, x acts as α. (so the resulting module is just V_α!)

Point: V_α does exist for all $\alpha \in \mathrm{End}_k(V)$.

What needs proof? We need to prove that the definition (*) satisfies the module axioms. (Try the proof!)

Next time: the proof; modules of quotients of $k[x]$.