AA2. Lecture 13.

02.11.2022.

Last time:

- induced modules: If M is a B-module and $\varphi: A \to B$ is an alg. hom, then M is automatically an A-module. $a \in A \xmapsto{\varphi} \varphi(a) \in B \circlearrowright M$
- Submodules (subgps closed under the algebra action)
- quotient (can be taken relative to any submodule)
- module hom (gp homs respecting the algebra action)

Today

- Iso Thms for modules
- External and internal direct sums
- Hw remark: Ex 1.22.4b) in [EH]: should assume that "A" is finite dim.

1. Iso. Thms.

Thm 1. Let R be a ring. Let $\varphi: M \to N$ be an R-module hom.

(1) The kernel $\ker\varphi := \{ m \in M \mid \varphi(m) = 0 \}$ is a submodule of M,

and the image $\operatorname{Im}\varphi := \{ \varphi(m) \mid m \in M \}$ is submodule of M.

(2) There is a well-defined module isomorphism $\bar{\varphi}: M/\ker\varphi \to \operatorname{Im}\varphi$

given by $\bar{\varphi}\left(m + \ker\varphi\right) = \varphi(m) \quad \forall m \in M$.

Pf: By gp theory we know $\ker\varphi$ is a subgp of M, $\operatorname{Im}\varphi$ is a subgp of N,

and $\bar{\varphi}$ is a well-defined gp iso. So for (1) it suffices to show that

$\ker\varphi$ and $\operatorname{im}\varphi$ are closed under the R-actions and for (2) it suffices

to check that $\bar{\varphi}$ is a module hom.

($R \cdot \ker\varphi \subseteq \ker\varphi$) Take $m \in \ker\varphi$ and $r \in R$. Then

$$\varphi(r \cdot m) \underset{\substack{\varphi \text{ is a} \\ \text{module} \\ \text{hom}}}{=\!=} r \cdot \varphi(m) \underset{m \in \ker\varphi}{=\!=} r \cdot 0 = 0$$

So $r \cdot m \in \ker\varphi$. We have proved that $\ker\varphi$ is closed under the R-action.

($R \cdot \operatorname{Im}\varphi \subseteq \operatorname{Im}\varphi$) Take $n \in \operatorname{Im}\varphi$ and $r \in R$. Then $n = \varphi(m)$ for some $m \in M$.

$$\text{So } r \cdot n = r \cdot \varphi(m) \underset{\substack{\varphi \text{ is a} \\ \text{mod. hom}}}{=\!=} \varphi(r \cdot m) \in \operatorname{Im}\varphi.$$

So $\operatorname{Im}\varphi$ is closed under the R-action.

($\bar{\varphi}$ respects the R-actions) Take $m \in M$ and $r \in R$. Then

φ is a mod. hom.

$$\bar{\varphi}(r \cdot (m + \ker\varphi)) \underset{\substack{\text{def of} \\ \text{quot. mod.}}}{=\!=} \bar{\varphi}(rm + \ker\varphi) = \varphi(r \cdot m) = r \cdot (\varphi(m)) = r \cdot (\bar{\varphi}(m + \ker\varphi)),$$

as desired $\square$

Thm 2. Let R be a ring, M an R-module, and $U, V \subseteq M$ submodules of M. Then the sum $U+V$ is an R-module, the intersection $U \cap V$ is an R-module, and we have $U/U\cap V \cong U+V/V$ as R-modules.

Thm 3. Let R be a ring, M an R-module, and $U \subseteq V \subseteq M$ be submodules of M. Then V/U is a submodule of M/U, and we have

$$\frac{M/U}{V/U} \cong M/V \text{ as } R\text{-modules.}$$

Pfs of Thms 2 & 3: Similar to that of Thm 1: apply the gp iso theorems after noting the relevant maps are module homs.

Prop. Let R be a ring and $\varphi: M \to N$ an R-module hom. Then for every
(2.26). submodule $W \subseteq N$ the preimage $\varphi^{-1}(W) := \{ m \in M \mid \varphi(m) \in W \}$
is an R-submodule of M and it contains the kernel of φ.

Pf: Ex.

Thm 4. (Correspondence thm) Let R be a ring, M an R-module, and $U \subseteq M$ an R-submod.
Then there are two mutually inverse inclusion-preserving bijections

$$\left\{ \begin{array}{c} \text{Submodules of } M \\ \text{containing } U \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{Submodules of} \\ M/U \end{array} \right\}$$

$$V \longmapsto V/U$$

$$\pi_U^{-1}(W) \longleftarrow\!\shortmid \ W$$

(where π_U is the proj $M \to M/U$, $m \mapsto m+U$)

Pf: Ex. (See Thm 2.28) $\square$

2. Direct sums of modules

As usual (as in gp/ring theory), there are two notions of direct sums for modules. Below, let R be a ring.

Def 1. (External direct sum, Def 2.17) Let $(M_i)_{i\in I}$ be a family of (not necessarily related) R-modules for some index set I. The set

$$\bigoplus_{i\in I}^{E} M_i = \{ \text{tuples } (m_i)_{i\in I} \mid m_i \in M_i \ \forall i, \ m_i \neq 0 \text{ only for finitely many } i\text{'s} \}$$

also often written as $\sum_{i\in I} m_i$ ← finite sum

is called the direct sum of $(M_i)_{i\in I}$.

Prop: (Easy) $\bigoplus_{i\in I} M_i$ is naturally an R-module in the above setting under the coordinatewise action $r\cdot(m_i)_{i\in I} = (rm_i)_{i\in I}$.

Def 2. (Internal direct sum, Def 2.15. (b)) Let M be an R-module.
We say M is the internal direct sum of a family $(M_i)_{i\in I}$ of submodules of M if (1) $M = \sum_{i\in I} M_i$, i.e., every elt in M is a sum of finitely elts from the submodules M_i. and (2). For every $j\in I$ we have $M_j \cap \sum_{i\neq j} M_i = 0$.
$\overset{\shortparallel}{\{0_M\}}$

Prop. (Connecting the two notions of direct sums)

Let M be an R-module and $(M_i)_{i\in I}$ a family of submodules of M.
Then the map $\bigoplus^E_{i\in I} M_i \longrightarrow M$ given by $(m_i)_{i\in I} \longmapsto \sum_{i\in I} m_i \in M$
is a module iso from $\bigoplus^E_{i\in I} M_i$ to M iff M is the internal direct sum of $(M_i)_{i\in M}$ in the sense of Def 2, ie, iff Conditions (1) and (2) hold.

Point of the Prop: We can use Conditions (1) and (2) to <u>recognize</u> when $M \cong \bigoplus_{i \in I}^{E} M_i$ (for submodules $(M_i)_{i \in I}$ of M).

Pf: next time, but try it yourself first!

Also next time:

modules of $k[t]$.