AA2. Lecture 12.

02.09.2022.

Last time:

- more examples of modules

$M_n(k) \overset{\text{mult.}}{\circlearrowright} k^n$ $\qquad$ $\mathrm{End}_k(V) \overset{\text{evaluation}}{\circlearrowright} V$

- modules vs representations. For any fixed k-alg. A, $\exists$ an equivalence

$\{A\text{-modules}\} \longleftrightarrow \{\text{representation of } A\}$

Thm 2.33. $\rightarrow$ (vec. space V with "nice" action $R \times M \to M$, $(r, m) \mapsto r \cdot m$) $\longmapsto$ (v.s. V & alg hom $A \to \mathrm{End}_k(V)$: V, $\varphi: r \mapsto (r\cdot)$, the action of r)

(module: V, action: $r \cdot m = \varphi(r)(m)$) $\longleftarrow$ (V, $\varphi: A \to \mathrm{End}(V)$)

Today. · an application of the rep. perspective · sub/quotient modules, homs.

1. An application of the rep. perspective

k: ground field

Prop: Let A, B be algebras and let V be a B-module. If $\varphi: A \to B$ is an alg. hom, then V is naturally an A-module with an action defined by

$$a \cdot v \overset{*}{=} \underbrace{\varphi(a) \cdot v}_{\text{the } B\text{-action}} \qquad \forall a \in A,\ v \in V$$

Special case:

If A is a subalg. of B & $\varphi: A \to B$ is just the inclusion hom $(a \mapsto a\ \forall a \in A)$ then the above action is just the action A inherits from B as mentioned before.

Pf: (Pf 1). Check the module axioms for $*$ directly.

(Pf 2). V is a B-module $\Rightarrow$ We have a rep (alg. hom) $B \to \mathrm{End}_k(V)$, $b \mapsto (b \cdot -)$

$\Rightarrow$ By concatenation we have an alg. hom $A \xrightarrow{\varphi} B \to \mathrm{End}_k(V)$, $a \mapsto \varphi(a) \mapsto (\varphi(a) \cdot -)$.

This hom/rep. corresponds to a module of A, which is given exactly by $*$. $\square$

3. Submodules and quotient modules

Def 1: Let R be a ring and M an R-module. A <u>submodule</u> of M is a subgp $U \leq M$ <u>closed under the R-action</u>, i.e., s.t. <u>$r \cdot u \in U \ \forall r \in R, u \in U$</u>.

Note: (1) Once the underlined condition holds, the module axioms must hold for U, so the underlined condition is equivalent to stipulating that U is an R-module in its own right.

(2) HW: If R is a k-algebra, then a submodule of M is automatically a subspace of M.

(3) Examples: (a) $R = k$, a k-mod $\equiv$ a k-v.s. V $\Rightarrow$ a submodule of V is precisely a subspace of V. (b) $M = R$, the regular module $\Rightarrow$ a submodule of R "=" an ideal of R.

Prop. (quotient modules) Let R be a ring, M an R-module, and N a submodule of N. Then the quotient gp M/N is naturally an R-module under the inherited action given by

$$R \times M/N \to M/N \qquad r.(m+N) \overset{*}{=} r.m + N.$$

pf: Well-definedness: We need to check that the r-action is well-defined $\forall r \in R$: if $m+N = m'+N$, then $m-m' \in N$, so $r\cdot m - r.m' \overset{\text{axiom}}{=} r.(\underbrace{m-m'}_{\in N}) \in N$ since N is a submodule. Therefore $r\cdot m+N = r\cdot m'+N$, as desired.

The action $*$ satisfies the module axioms: EX. $\square$

Examples:

(1) $R = \mathbb{Z}$, $M = \mathbb{Z} = R$. the regular module.

Then $\forall d \in \mathbb{Z}$, $\langle d \rangle := \{ nd : n \in \mathbb{Z} \}$ is an ideal/a submodule of R.

The gp quotient $\mathbb{Z}/\langle d \rangle$ is given by the classes

$$\mathbb{Z}/\langle d \rangle = \{ \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \dots, \overline{d-1} \} \text{ by gp theory.}$$

(e.g. $\bar{2} + \bar{1} = \bar{3}$ and $\bar{3} + \bar{4} = \bar{7} = \bar{2}$ if $d=5$)

$\bar{2}$ = $2 + \langle d \rangle$

But $\mathbb{Z}/\langle d \rangle$ is also a module of R, with the action

$$a \cdot \bar{j} = \overline{aj}$$

(e.g. if $d=5$, then $4 \cdot \bar{3} = \overline{4 \cdot 3} = \overline{12} = \bar{2}$.)

3. Module homomorphisms

Next time: Iso Thms. Simple modules

Def: Let R be a ring and M, N be R-modules. A map $\varphi: M \to N$ is called an <u>R-module homomorphism</u> if

(i) φ is a gp hom, i.e., $\varphi(m+m') = \varphi(m) + \varphi(m')$ $\forall m, m' \in M$

and (ii) φ respects the R-action, i.e., $\varphi(r \cdot m) = r \cdot \varphi(m)$ $\forall r \in R, m \in M$

(Again: If R a k-algebra then (i) + (ii) imply that an R-module hom. is automatically a k-linear map.)

A <u>module isomorphism</u> is a bijection module hom.

Eg: If M is an R-module and N an module of M, then

(a) The natural inclusion $\iota_N : N \to M$, $n \mapsto n$ $\forall n \in N$ is an R-module hom;

(b) The natural projection $\pi_N : M \to M/N$, $m \mapsto m+N$ $\forall m \in M$ is an R-module hom.