AA2. Lecture 12

02.07.2022.

Last time:

- Ideals in $k[t]$ are all principal.
- Def of modules: A module over a ring R is an abelian gp $(M,+)$ equipped with a map $R\times M \to M$, $(r,m)\mapsto r\cdot m$ $\forall r\in R, m\in M$ s.t.
  - $r\cdot(m+n) = r\cdot m + r\cdot n$
  - $(r+s)\cdot m = r\cdot m + s\cdot m$, $\underbrace{(rs)\cdot m = r\cdot(s\cdot m)}$, $1\cdot m = m$ $\Big\}$ $\forall r,s\in R$, $m,n\in M$

  becomes "$m\cdot(rs) = (m\cdot r)\cdot s$" in the def. of right modules
- Examples of modules:

  k-modules $\equiv$ k-vec spaces, $\mathbb{Z}$-modules $\equiv$ Abelian gps

  regular module of R = $R \circlearrowleft R$, ideals of a ring R are R-modules

Today:

- more examples
- another def
- Submodules and quotient modules

1. More examples of modules

(5) natural modules of matrix algebras $A = M_n(k) \overset{\text{left mult}}{\circlearrowright} V := k^n$ $\quad X \cdot v := Xv$

$\qquad\quad \| $

(i) By properties of matrix-vec. mult. (e.g. $X(v+w) = Xv + Xw$), $\left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in k \ \forall i \right\}$

it's easy to see that the left mult. action makes k^n an A-module.

Note*: (left vs. right) Note that the above action would not make k^n a right A-module because in general $(XY)v = Y(Xv)$ in matrix multiplications.

Instead, the set $V' := \{ [a_1 \cdots a_n] : a_i \in k \ \forall i \}$ can be made a right A-module if we define the action to be right mult: $\underset{\underset{V'}{\in}}{v'} \cdot \underset{\underset{A}{\in}}{X} = (v'X)$.

(ii) Any subalgebra of $A = M_n(k)$ also acts on V by the same (inherited) action.

e.g. $T_n(k)$ _ the upper Δ matrices

(b) natural module of endomorphism algebras $\quad A = \mathrm{End}_k(V) \circlearrowright V$

(i) The map $\mathrm{End}_k(V) \times V \to V$ defined by $\qquad f : V \to V$

$$(f, v) \mapsto f \cdot v := f(v)$$

→ the action is just "evaluation"

makes V an A-module. $\quad$ Ex: Check the axioms.

(e.g. we'd need

$f \cdot (v+w) = f \cdot v + f \cdot w,$

i.e., $f(v+w) = f(v) + f(w)$.

this holds by the linearity of f.)

(ii) As in the last example, subalgebras of A also inherit as a module with the same action.

In fact, every module V of an algebra corresponds to an action (by evaluation) of some subalgebra of $\mathrm{End}_k(V)$ on V in the sense of (b).(ii).

More precisely ...

2. Modules $\equiv$ Representations

k: ground field over which things are done

- Let V be an module over/of an algebra A ($A \circlearrowright V$).

Recall that $B := \mathrm{End}_k(V)$ is an algebra with mult. being composition.

Ex: - The axioms for modules implies that $\forall a \in A$, the map $\underline{\varphi(a)} : V \to V$ "the action of a"

given by $\big(\varphi(a)\big)(v) := a \cdot v \in V$

is a linear map, so $\varphi(a) \in \mathrm{End}_k(V)$, yielding a map $\varphi : A \to \mathrm{End}_k(V)$
$a \mapsto \varphi(a)$,

- The axioms also imply that φ is an algebra hom.

Thus, the action $A \circlearrowright V$ corresponds to an algebra hom $\varphi : A \to \mathrm{End}_k(V)$

and to the evaluation action $\varphi(A) := \mathrm{Im}(A) \circlearrowright V$

We call the data (V, φ) a representation of A.

Conversely, given a rep. of A, i.e., given a v.s V and an alg hom $\varphi : A \to \operatorname{End}_k(V)$, we can make V an A-module by defining the action $A \times V \to V$ by

$$a \cdot v = \varphi(a)(v) \qquad \forall a \in A, v \in V$$

Ex: The action defined above satisfies the module axioms, as in Eg.(b).

(e.g. "$a \cdot (v+w) = a \cdot v + a \cdot w$" because $\varphi(a)$ is linear

$(a+b) \cdot v = a \cdot v + b \cdot v$, i.e., $\varphi(a+b)(v) = \varphi(a)(v) + \varphi(b)(v)$

$= (\varphi(a) + \varphi(b))(v)$

because φ is a hom.)

Conclusion: Modules $\equiv$ Representations.

E.g. **Fact:** The gp alg $A = kS_3$ has the v.s $V = k^3$ as a module, with an action s.t.

basis $\{e_1, e_2, e_3\}$

$$\pi \cdot e_i = e_{\pi(i)}$$

e.g. $\quad (12)\, e_1 = e_2$

This action corresponds to the rep $A \to \mathrm{End}_k(V) = M_3(k)$ with

$$e_{S_3} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (12) \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (23) \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(123) \longmapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (132) \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad (13) \longmapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next time: Submodules, quotient modules, module homs.