AA2. Lecture 10.

02.02.2022.

Last time:

- examples of algebra isos
- principal ideals

Prop 1: Every ideal I in the polynomial algebra $k[t]$ is principal.

Today:

- pf and applications
- start Ch 2: Modules of algebras

  definition, remarks, and examples

1. Prop 1.

Pf: Let p be any elt in I of minimal degree. We claim that $I = \langle p \rangle$, i.e., every elt in f is a multiple of p. To see this, consider dividing f by p. Then we have

$$f = q \cdot p + r \quad \text{for some } r \in k[t] \text{ with } \deg(r) < \deg(p) \text{ or } r = 0.$$

Now, since $f, p \in I$, $r = f - qp \in I$. By the minimality assumption, it follows that $r = 0$, so $f = qp \in \langle p \rangle$, as desired. $\square$

An Application: (E.g.(6) from last time) The kernel of the eval. hom

$$\mathrm{Eval}_i : \mathbb{R}[x] \to \mathbb{C}, \quad x \mapsto i$$

is an ideal, hence a principal ideal, of $\mathbb{R}[x]$. Moreover, by the last proof, the kernel is gen'd by any min-deg. elt in it.

It is <u>easy</u> to see that no nonzero elt of deg. less than 2 is in the kernel while $x^2+1 \in \ker \mathrm{Eval}_i$, so $\ker(\mathrm{Eval}_i) = \langle x^2+1 \rangle$.

$(ax+b \mapsto ai+b \neq 0$ if $ax+b \neq 0)$ $\quad (x^2+1 \mapsto i^2+1 = -1+1 = 0)$

Thus, $\mathbb{R}[x]/\langle x^2+1 \rangle \cong \mathrm{Im}(\mathrm{Eval}_i) = \mathbb{C}$.

Ex: Mimic the argument to prove E.g.(3) from last time.

2. Modules of rings and algebras

k: a field.

Def: A (left) module over/of a ring R is an abelian gp $(M, +)$

we'll almost always only talk about left module

with an action of R, i.e., a map $R \times M \to M$, w/ the following properties:

$$(r, m) \mapsto r \cdot m$$

- $\forall r \in R$, $r \cdot (m+n) = r \cdot m + r \cdot n$ (the action of r respects addition in M, i.e., is a hom from M to M)
- $\forall r, s \in R$, $m \in M$,

$$\left.\begin{aligned} (r+s) \cdot m &= r \cdot m + s \cdot m \\ r \cdot (s \cdot m) &= (rs) \cdot m \\ 1_R \cdot m &= m \end{aligned}\right\}$$

the map $r \mapsto \underbrace{(r \cdot -)}_{\text{the action of } r \text{ on } M}$ should respect addition, multiplication, and unit.

Remarks: (Module theory generalize linear algebra.)

(1) Modules of rings generalize vector spaces of fields:

if $R = k$, a field, then an R-module is just an abelian gp $(M,+)$ with a k-action s.t. the four axioms are satisfied

$\forall c,d \in k, m,n \in M$, $c\cdot(m+n) = c\cdot m + c\cdot n$, $(c+d)\cdot m = c\cdot m + d\cdot m$, $(cd)\cdot m = c(d\cdot m)$, $1\cdot m = m$.

That is, a k-module is precisely a k-vector space, with the scaling map being the action.

(2) Modules of k-algebras are automatically k-vector spaces:

(Lemma 2.5.)

they are rings

Let A be a k-algebra. The scaling map on an A-module M can be defined via the embedding $\iota: k \to A$, $\lambda \mapsto \lambda\cdot 1_A$, i.e., by defining

$$\lambda\cdot m := \iota(\lambda)\cdot m = (\lambda\cdot 1_A)\cdot m. \ \forall \lambda\in k, m\in M.$$

familiar from before.

Ex. Prove: this scaling map makes M a k-v.s.

Examples of modules

(1) Again: k-vector spaces $\equiv$ k-modules.

(2) $\mathbb{Z}$-modules:

- If the ring R is $\mathbb{Z}$, then an R-module is certainly an abelian gp.
- Given any abelian gp $(M, +)$, we may define a map $\mathbb{Z} \times M \to M$ by

in this sense,

$\mathbb{Z}$-modules are the "same" as abelian gps

$$n \cdot a = \begin{cases} 0 & \text{if } n = 0 \\ \underbrace{a + \cdots + a}_{n \text{ copies}} & \text{if } n > 0 \\ -[\underbrace{(-n)}_{>0} \cdot a] & \text{if } n < 0 \end{cases} \qquad \forall\, n \in \mathbb{Z},\ a \in M$$

Ex: Prove that this map makes M an $\mathbb{Z}$-module.

(3) regular modules: Every ring is a left module of itself, with the action being left multiplication:

$$\forall r \in R,\ m \in R, \qquad r \cdot m = rm, \text{ the product in } R.$$

Ex: Check that the action satisfies the axioms.

(e.g. $r\cdot(s\cdot m)=(rs)\cdot m$ is $r(sm)=(rs)m$, which holds by associativity of mult.)

(4) ideals: Every left ideal I of a ring R is naturally a module of R, with elts of R acting on I by left multiplication.

Ex: check the claim

(the absorption property $RI \subseteq I$ ensures the action map $R \times I \to I$ makes sense).

Next time: · more examples
· an equiv. def.
· more basic notions.