Math 4140/5140. Abstract Algebra 2.

Jan 10. 2022.

Course information:

Instructor: Tianyuan Xu (Eddy). Math 202.

Topics: Algebras and representation theory

Website: https://math.colorado.edu/~tixu6187/s22aa2.html

— lecture notes, hw and announcements under "LECTURES"

— other syllabus info on the website too, including the Canvas link.

Office Hours: Zoom appointments.

Grading: HW 30%, Midterm 30%, Final Exam 40%

HW:
- Posted weekly on Wednesdays on the website
- Due in Canvas/Assignments the next Wed at 11:59 pm.
  the deadline is strict, and a single pdf should be submitted for each assignment
- The first assignment will be posted on Jan. 12.

Textbook: Algebras and Representation Theory, by Erdmann and Holm.
(in Canvas/Files and on the website)

Today: Review of familiar notions: axiomatic def. of groups and rings.
basic facts from group theory.

1. Groups.

Def. A <u>group</u> (gp) is a pair $(G, \cdot)$ where G is a nonempty set and $\cdot$ is an operation such that (s.t.) the following holds:

(0). (Closure) The operation $\cdot$ is a map $\cdot : G \times G \to G$, ie, for all $a, b \in G$, there is an element (elt) $a \cdot b$ in G.

(1). (Associativity) $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(2). (Identity) There is an elt e in G s.t. $a \cdot e = a = e \cdot a$ $\forall a \in G$.

(3). (Inverse) $\forall a \in G$, there is an elt $b \in G$ s.t. $a \cdot b = e = b \cdot a$.

We say a gp G is <u>abelian</u> if $a \cdot b = b \cdot a$ $\forall a, b \in G$.

Examples :

(1) (Some number systems)

$(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ form abelian gps.

the integers — the rational numbers.

$(\mathbb{N}, +)$ is not a gp, since the inverse axiom fails

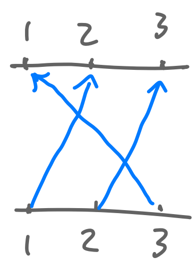
for this course, $\mathbb{N}$ is always equal to $\mathbb{N} = \mathbb{Z}_{\geq 0}$

(2) The symmetric group $S_n = \{$ bijections from $[n] = \{1, 2, \cdots, n\}$ to $[n]$ $\}$ is a gp under composition for each $n \geq 1$.

called "permutations"

e.g.

$$\begin{pmatrix} f : [3] \to [3] \\ 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{pmatrix} =$$



$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \underline{2\,3\,1} \in S_3.$$

"one-line notation"

Recall that the identity of S_n is the identity bijection $e = \begin{bmatrix} 1 & 2 & 3 & \cdots \\ 1 & 2 & 3 & \cdots \end{bmatrix}$

S_4. $f = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{bmatrix}$ $g = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$

$\Downarrow$

$$f \cdot g = f \circ g = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

f has inverse $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{bmatrix}$.

We'll assume familiarity with the basic notions, facts, and proof techniques from gp theory. e.g. subgps. Lagrange's Thm. gp homomorphisms. Isomorphism thms, gp actions, ...

Selected notions / ideas / results:

· A routine proof ("uniqueness of inverse"):

Prop: Each elt g in a gp G has a unique inverse.

Pf: Say g_1 and g_2 are two inverses of g. We just need to show that $g_1 = g_2$.

Here's how:

$$g_1 \underset{\text{id. axiom}}{=} g_1 e = g_1 (g g_2) \overset{\text{associativity}}{=} (g_1 g) g_2 \underset{\text{id. axiom}}{=} e g_2 = g_2 .$$

$\square$

· A "natural" definition:

Next time: more review for gps, rings and lin. algebra

Def. A <u>group homomorphism</u> is a map $f: G \to H$ between two gps G and H s.t.

(1) $f(e_G) = e_H$

and (2) $f(g_1 g_2) = f(g_1) f(g_2)$ $\forall g_1, g_2 \in G$.

"f respects the gp operations"

E.g. The set $GL_n(\mathbb{R}) = \{ n\times n \text{ invertible matrices over } \mathbb{R} \}$ is a gp under mult, and so is $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. We have $\det(I_n) = 1$, $\det(AB) = \det(A)\det(B)$, so $\det: GL_n \to \mathbb{R}^\times$ is a gp hom.