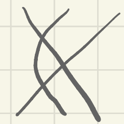
Coxeter diagram

S_3 : permutations of $\{1,2,3\}$

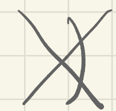
$S = \{s_1, s_2\}$, $s_i = (i \quad i+1)$

Relations:

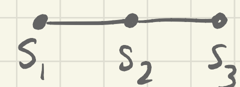
- $s_1^2 = s_2^2 = 1 = id$
- $s_1 s_2 s_1 = s_2 s_1 s_2$

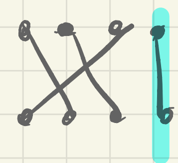


or



S_4 : permutations of $\{1,2,3,4\}$



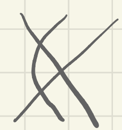


$S = \{s_1, s_2, s_3\}$, $s_i = (i \quad i+1)$

Relations:

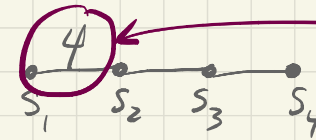
- $s_1^2 = s_2^2 = s_3^2 = 1 = id$
- $s_1 s_2 s_1 = s_2 s_1 s_2$
- $s_2 s_3 s_2 = s_3 s_2 s_3$

$s_1 s_3 = s_3 s_1$



or

B_4 Generating set: $S = \{s_1, s_2, s_3, s_4\}$



Relations: • $s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1 = id$

• Commuting generators:

$$s_1 s_4 = s_4 s_1 \qquad s_2 s_4 = s_4 s_2$$
$$s_1 s_3 = s_3 s_1$$

• Braid Relations

$$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$$

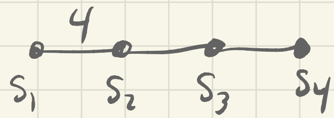
length here comes from

$$s_2 s_3 s_2 = s_3 s_2 s_3$$
$$s_3 s_4 s_3 = s_4 s_3 s_4$$

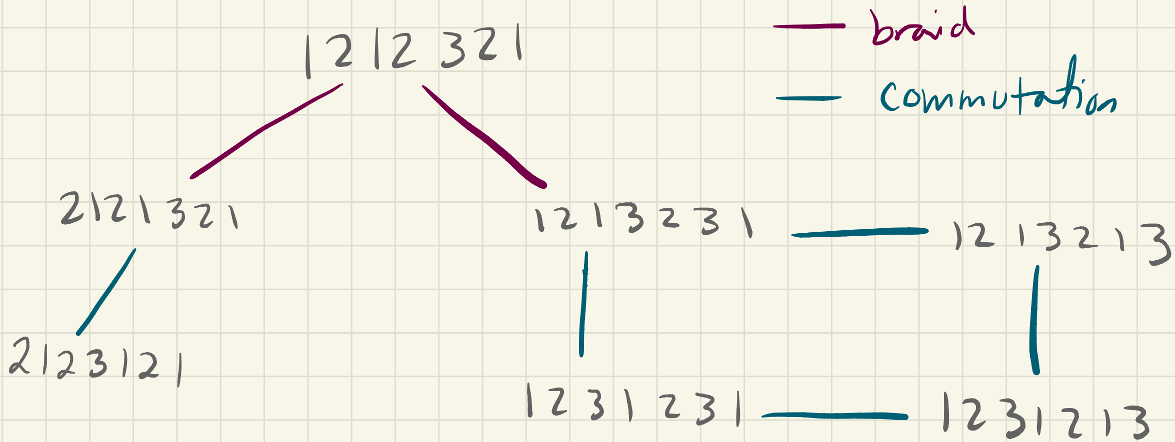
Acts like rearrangings coins and flipping them over.

$$s_1 s_2 s_1 s_2 \,(H_1 \;\; H_2) = s_1 s_2 s_1 \,(H2 \;\; H1) = s_1 s_2 \,(T2 \;\; H1)$$
$$= s_1 \,(H1 \;\; T2)$$
$$= T1 \;\; T2$$

B_4



Draw the reduced word graph of $w = s_1 s_2 s_1 s_2 s_3 s_2 s_1 = 1212321$



"Matsumoto's Theorem" Can move from any reduced expression of a fixed $w \in W$ to any other using only commutations and braid relations.

Fully Commutative Elements:

Def: If for any two reduced expressions $\bar{w}_1$ and $\bar{w}_2$ for $w \in W$, we can obtain one from the other using only commutations, then we say w is <u>fully commutative</u>.

Criterion: (Stembridge)

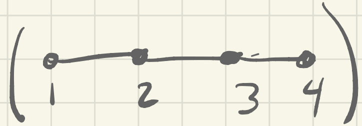
An element $w \in W$ is fully commutative if and only if no reduced expression for w contains an opportunity to apply a <u>braid move</u>

depends on the length of the relation

Q How can we tell if an element is fully commutative (FC)?

- The def. suggests we draw the full reduced word graph.
- The criterion suggests checking <u>all</u> reduced expressions.
- Use heaps!
  - Define via examples!

Heaps!

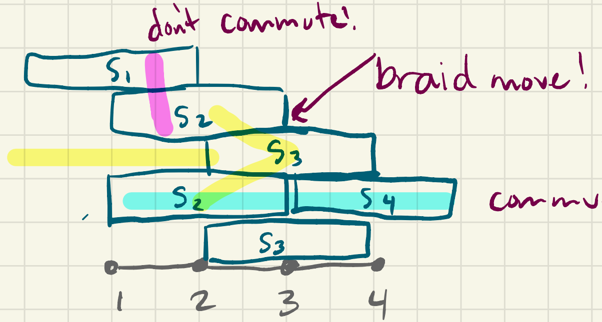
Example 1: S_5 ()

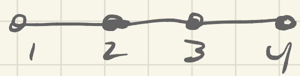
$s_1 s_2 s_1 = s_2 s_1 s_2$

$s_2 s_3 s_2 = s_3 s_2 s_3$

$s_3 s_4 s_3 = s_4 s_3 s_4$

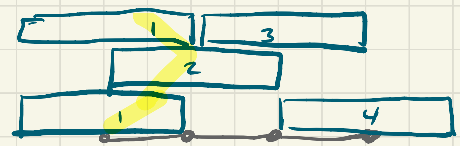
$w = s_1 s_2 s_3 s_4 s_2 s_3$ (not FC)



Example 2: S_5 

$w = s_1 s_3 s_2 s_4 s_1$

Not FC!

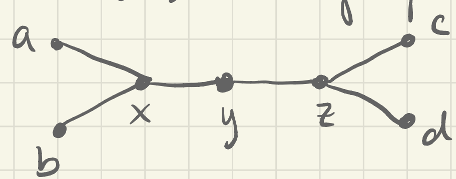


To-do for next Thursday (2/6/2020)

- In the symmetric group, S_5, draw the heaps for the following words:
  - $s_1 s_3 s_2 s_1 s_4 s_3$
  - $s_4 s_2 s_1 s_3 s_2$
  - $s_3 s_1 s_4 s_3 s_1 s_2$
  - $s_3 s_2 s_4 s_1 s_2 s_3$
  - Are any of these words equal in S_4?
  - Are any of these words reduced?
  - Do any of these words represent fully commutative elements?
- In either S_4 or type B_4, find an element with at least 3 heaps!
- What properties of the heap of a word show that it is the heap of a fully commutative element?

  (look at Stembridge's criterion)

- For thought: Let W be the group with the diagram



(
- vertices are generators that square to the identity;
- edges indicate relations of the form $121 = 212$;
- vertices not connected by an edge commute
)

- Tell me whatever you can about the element represented by the word

$$w = abxyzcdzyxabx$$

(Is it reduced? What might the heap look like? Is it FC?

What else do you want to know?)