Last time:

- Reduced word graph of an elt w

  vertices: the red. words of w

  edges: two words are connected if they are related by a single braid move

- Matsumoto's Thm: this graph is connected.

A variation:

Color the edges by the braid pair. e.g. $sts = tst$ $\{s, t\}$

Even, single out commutation moves $st = ts$ by a distinguished color, say red.

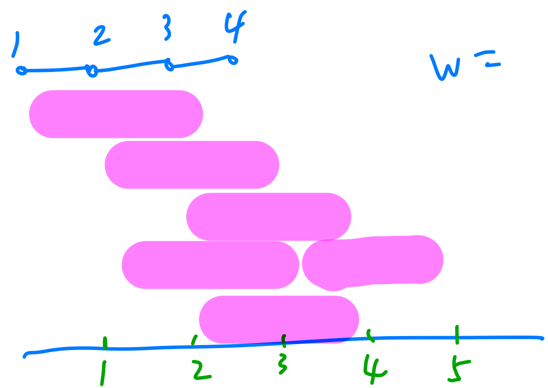
— FC elts: def. Stembridge's word criterion

(w is FC $\iff$ no red. word of w contains a longer braid)

(Equiv: the red word graph of w is 'red-connected',
i.e., $\exists!$ red-connected component.

commutation classes

— Heaps e.g.

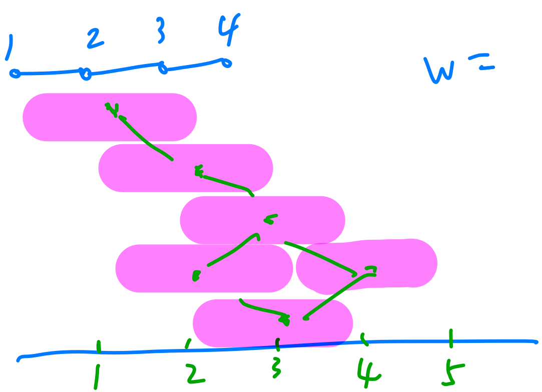


$w = s_1 s_2 s_3 s_4 s_2 s_3$

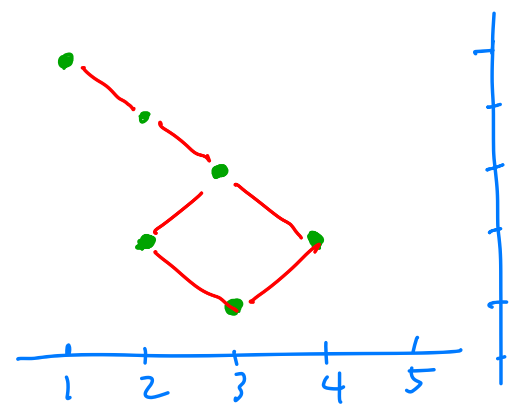
Today.

① Another way to draw heap:

- generators fall as points instead of bricks
- physical blocking of the bricks become edges

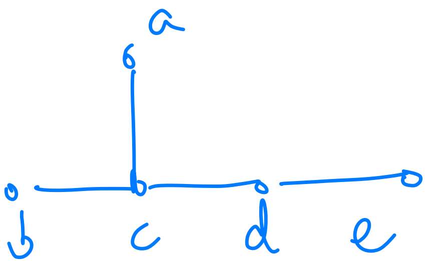


$w = s_1 s_2 s_3 s_4 s_2 s_3$

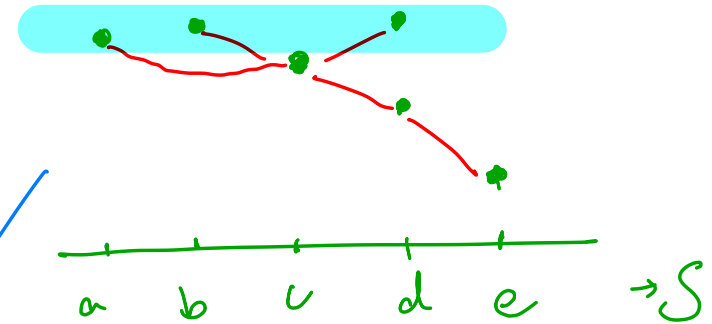


slight advantage: easier for 'non-linear' graphs

e.g.



$w = abdcde$

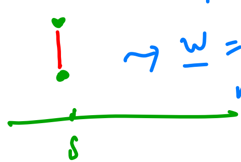


These graphs will be the 'Hasse diagram of the heap poset'

② So given a Coxeter gp (i.e. its diagram) and a word $\underline{w}$ on the generators, we can draw the heap of the word.

Turns out: we can tell whether $\underline{w}$ is a red. word of an FC elt by looking at the heap $H(\underline{w})$

Prop (Stembridge's Heap Criterion for FC) $\underline{w}$ is the reduced word of an FC elt $\iff$ in $H(\underline{w})$,



$\rightsquigarrow \underline{w} = \dots(s\,s)\dots$ not red! $\longleftrightarrow$

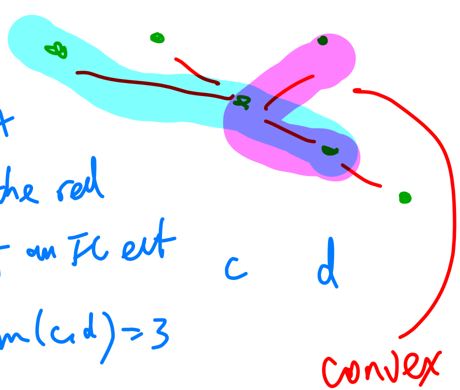
(1) No column $s \in S$ has two points connected by an edge

(2) There is no 'convex chain' $sts\dots$ of length $m(s,t)$ where $m(s,t) \geq 3$.

'Convex chain' : <u>chain</u>: points $v_1, v_2, \ldots, v_n$ s.t. $v_i v_{i+1}$ is connected by an edge $\forall i$.

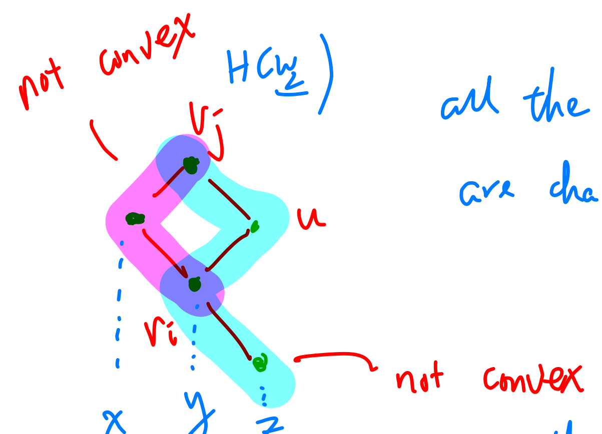
<u>Convex chain</u>: $\forall i, j \nexists u$ s.t. $v_i - u - v_j$

H($\underline{w_1}$)



$\underline{w_1}$ is not the red word of an FC elt if $m(c,d)=3$

H($\underline{w_2}$)



all the highlighted parts are chains

w_2 is the red word of an FC elt if $m(x,y)=3$, $m(y,z)=4$

Also recall that all reduced words of an FC elt give us the same heap picture.

$m(b,d)=2$

w is FC $\Longrightarrow$ $w = \ldots bd(\ldots) = \ldots db(\ldots)$

Points:

- Given an FC elt, we can associate to it a unique heap picture independent of the choice of red words
- Given a word, its heap picture allows us to tell if the word is FC.

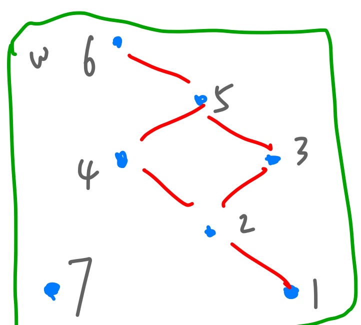
③ The n-value of an FC elt.

w FC elt $\longrightarrow$ Heap $H(w)$, a picture.

Def. $n(w) = \max\{\, |A| : A \text{ is an antichain in } H(w) \,\}$

an antichain: a subset of the points A s.t. no two points are connected by a chain.

e.g.



$\{2,5\}$ is not an antichain

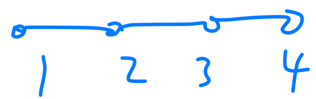
$\{1,6\}$ not an antichain

$\{7,1\}$, $\{6,7\}$, $\{u,7 : u \neq 7\}$, $\{3,4\}$, $\{3,4,7\}$ are antichains

In fact, $n(w) = 3$.

<u>Hw:</u> Review the definitions of a partially ordered set (poset), Covering relations in posets, Hasse diagrams.

<u>Sarah's Exercise:</u>

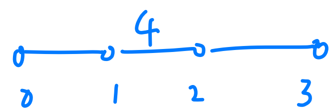
$S_5 = A_4$ 

Draw the heaps of $w_1 = s_3 s_1 s_4 s_1 s_3 s_2$, $w_2 = s_3 s_4 s_1 s_2 s_3$, $w_3 = s_4 s_2 s_1 s_3 s_2$. Are they reduced? <u>FC?</u> n-value of FC?

<u>Exercise:</u> Suppose w is the red word of a FC elt. Then w has a unique red word $\Longleftrightarrow$ $n(w) = 1$.

Questions

– Given (the presentation of) a Coxeter gp W, can we

(example: 'F4' :  ~ $\begin{array}{cccc|c} 0 & 1 & 2 & 3 & \\ \hline 1 & 3 & 2 & 2 & 0 \\ 3 & 1 & 4 & 2 & 1 \\ 2 & 4 & 1 & 3 & 2 \\ 2 & 2 & 3 & 1 & 3 \end{array}$

↳ m-matrix)

* : count the FC elements of n-value 2 ?

✓ (a). Determine if W is finite ?

(b). enumerate all elts of W by length, regardless of whether W is finite or not ?

(if we want to repr. an elt w by a red word, is there a canonical word ?) ⟵ related / likely necessary : given $w \in W$, determine if w is FC.

√ (c) Determine if W is FC-finite ?

(d) enumerate all FC elts of W by length, FC finite or not ?

(e) enumerate all FC elt of w by n-value, FC finite or not?

(fact: there are Coxeter gps with inf. many FC elts but only finitely FC elt of n-value 2.)

√ : can use known results to cheat.

To do: check what's available on SageMath for red. words.