MATH 8174. HOMEWORK 8

(You do not have to hand this in.)

Note: [Hum] = Humphreys, [EW] = Erdmann–Wildon

(1) Read Sections 21.2 and 21.4 of [Hum].

(Fix an algebraically closed ground field k of characteristic 0 below.)

(2) Let (C, Δ, ε) be a coalgebra (over k). Show that the dual space $A := C^*$ is naturally an algebra (describe the multiplication and unit for A first).

(3) Let $(B, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra.
   (a) Show that if B admits an antipode making it a Hopf algebra, then the antipode is unique.
   (b) Show that if B is cocommutative, then for any B-modules V, W, the natural isomorphism of vector spaces $\varphi : V \otimes W \to W \otimes V, w \otimes v \mapsto v \otimes w$ is an isomorphism of B-modules.

(4) Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra and let $\mathrm{rep}(H)$ be the category of finite dimensional left H-modules. Let V be an object in $\mathrm{rep}(H)$, i.e., let V be a finite dimensional H-module. In class we described how to make the dual space V^* an H-module by using the antipode S. Consequently, the tensors $V^* \otimes V$ and $V \otimes V^*$ are H-modules. Now consider the linear maps

$$\mathrm{ev}_V : V^* \otimes V \to k, f \otimes v \mapsto f(v)$$

and

$$\mathrm{coev}_V : k \to V \otimes V^*, 1 \mapsto \sum_{i=1}^{n} v_i \otimes \hat{v}_i,$$

where $\{v_1, \ldots, v_n\}$ is a basis of V and $\{\hat{v}_1, \ldots, \hat{v}_n\}$ is the corresponding basis of V^* with $\hat{v}_j(v_i) = \delta_{ij}$. Prove that ev_V and coev_V are both homomorphisms of H-modules, then show that

$$(\mathrm{id}_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$

and

$$(\mathrm{ev}_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes \mathrm{coev}_V) = \mathrm{id}_{V^*}.$$

(The equalities imply that V^* is a *left dual* of V in $\mathrm{rep}(H)$ in a categorical sense.)

(5) Prove the fact that "$\Lambda^j(V) \cong L(\omega_j)$ for all $1 \leq j < n$" in type A_n from the notes of December 7th. Do we similarly have $V \cong L(\omega_1)$ for the natural module V and fundamental weight ω_1 in type B_n?