MATH 8174. HOMEWORK 7
Due Monday, November 30

Note: [Hum] = Humphreys, [EW] = Erdmann–Wildon

(1) Read Chapter 18 of [Hum].

(2) Let L be a Lie algebra and let $\varphi : L \to L$ be an automorphism of L. Prove that φ respects the Killing form κ in the sense that $\kappa(\varphi(x), \varphi(y)) = \kappa(x, y)$ for all $x, y \in L$. This fact was used on the last page of our notes for November 11.

(3) Let $L = H \oplus (\oplus_{\alpha\in\Phi} L_\alpha)$ be the Cartan decomposition of a complex semisimple Lie algebra L with respect to a Cartan subalgebra H. Let Δ be a base of the root system Φ. For each $\alpha \in \Phi$, fix a nonzero element $e_\alpha \in L_\alpha$ and let $\{e_\alpha, f_\alpha, h_\alpha\}$ be the corresponding $\mathfrak{sl}_2$ triple in L (so $y_\alpha \in L_{-\alpha}$ and $h_\alpha \in H$).

  (a) Show that if φ is an automorphism of L such that $\varphi(e_\alpha) = -f_\alpha, \varphi(f_\alpha) = -e_\alpha$ for some $\alpha \in \Phi$, then $\varphi(h_\alpha) = -h_\alpha$.

  (b) Prove that there exists an automorphism of L such that $\varphi(e_\alpha) = -f_\alpha, \varphi(f_\alpha) = -e_\alpha$ and $\varphi(h_\alpha) = -h_\alpha$ for all $\alpha \in \Delta$.

  (c) Let φ be an automorphism satsifying the conditions in (b). Is is necessarily true that $\varphi(e_\alpha) = -f_\alpha, \varphi(f_\alpha) = -e_\alpha$ for all $\alpha \in \Phi$ and $\varphi(h) = -h$ for all $h \in H$? Explain your reasoning.

(4) Let L be an abelian Lie algebra over a field k, say with basis $\{x_1, \dots, x_n\}$. What is the universal enveloping algebra of L like?

(5) Prove that the PBW basis theorem implies the other version of the PBW theorem ([Hum] Theorem 17.3, i.e., our "Theorem 1").

(6) Let k be a field of characteristic 0. The *first Weyl algebra over* k is the algebra $A_1 = A_1(k) := k\langle x, y\rangle/\langle yx = xy + 1\rangle$. Prove that the set $\beta = \{x^m y^n : m, n \in \mathbb{Z}_{\geq 0}\}$ form a basis of A_1 in the following two ways.

  (a) First show that β spans A_1. Then explain why we can make the polynomial ring $V = k[t]$ a (left) module of A by letting x and y act linearly by $x \cdot f = t \cdot f$ for all $f \in V$ and $y \cdot t^n = nt^{n-1}$ for all $n \geq 0$ (so y acts as differentiation with respect to t if we view f as a function). Finally, use V to show that β is linearly independent (and hence a basis).

  (b) Use Bergman's diamond lemma to show that β is a basis of β. Make sure you explain the setup clearly.