So far. (1) Claim: The map

$$\mathcal{HW}: \left\{ \begin{matrix} \text{f.d. irreps} \\ \text{of } \underline{L} \\ \text{s.s.} \end{matrix} \right\} \Big/ \cong \longrightarrow P^+ \subset \bigoplus_{i=1}^{\text{rk } L} \mathbb{N}\omega_i .$$

$$L \longmapsto \text{the h.w. in } L.$$

(2) Defined the Verma module $M(\lambda)$ for any $\lambda \in H^*$ by

$$M(\lambda) = \underset{\mathcal{U}(L)}{\mathcal{U}(L)} \underset{\mathcal{U}(B)}{\otimes} \underset{\mathcal{U}(B)}{\mathbb{C}_\lambda} \sim \text{1-d} =: \mathcal{U}(L) \underset{\mathcal{U}(B)}{\otimes} \mathbb{C}_\lambda .$$

$B = N^+ \oplus H$

$h \cdot = \lambda(h)$

$N^+ \cdot = 0$

Today: $M(\lambda)$ has a unique maximal proper submodule / and hence a unique nontrivial irr. quotient $L(\lambda)$, and $M(\lambda)$ is universal among h.w. L-modules with wt. λ $\rightsquigarrow$ ($\lambda \in P^+ \mapsto L(\lambda)$ is the inverse of $\mathcal{HW}$.)

Preparation: $M(\lambda)$ as a h.w. module. Recall that $\forall x \in L$

$$x \cdot \left(1_{U(L)} \otimes 1_\lambda\right) = (x) \otimes 1_\lambda = 1_{U(L)} \otimes (x \cdot 1_\lambda) \quad \text{if } x \in U(B)$$

$$= 1_{U(L)} \otimes \underline{\underline{\lambda(x)}}\, 1_\lambda$$

$$= \begin{cases} \lambda(x) \left(1_{U(L)} \otimes 1_\lambda \right). & \text{if } x \in H \\ 0 & \text{if } x \in N^+. \end{cases}$$

So the vector $v_+ := 1_{U(L)} \otimes 1_\lambda$ is a h.w. vector and $M(\lambda)$ is a h.w. module.

(in particular, if $x = e_\alpha$, $\alpha \in \Phi^+$)

Prop: Let $\lambda \in H^*$ and let V be a h.w. module of L w/ h.w. λ.

(recall that V must have a unique h.w. vector v_+ up to scalar.)

Then (a). every submodule W of V is a direct sum of wt spaces.

(Compatibility w/. $V = \oplus V_\lambda$.)

(b). V is indecomposable and has a unique maximal proper submodule

Rad V and hence has a unique irreducible quotient.

Pf: We first prove (a) $\Rightarrow$ (b): take any proper submodule W of V. Then $W = \bigoplus_{\mu \in H^*} W_\mu$ by (a)

where necessarily $W_\lambda = 0$ since otherwise $V_\lambda \subseteq W$ would generate V, forcing

$W = V$. Now take $\operatorname{Rad} V = \sum W$, summed over all proper submodules. We still have

$\operatorname{Rad} V \cap V_\lambda = 0$. and $\operatorname{Rad} V$ is clearly the unique max proper submod. of V.

It remains to prove (a). Let $W \le V$ be a submodule. Let

$w = v_1 + v_2 + \cdots + v_k$, where the decomposition is obtained from $V = \bigoplus V_\mu$

and $v_i \in V_{\mu_i}$, μ_i a wt, for each i. We need to show $v_i \in W$ for

all $1 \le i \le k$. If not, find w with k minimal. Then $k > 1$, and $v_j \notin W$

for all $1 \le j \le k$ (otherwise $w - v_j$ has fewer parts and is in W).

Take $h \in H$ s.t. $\mu_1(h) \ne \mu_2(h)$, then $(h - \mu_1(h) \cdot 1) \cdot w = \sum_{j=2}^{k} \big(\mu_j(h) - \mu_1(h)\big) v_j$.

The sum on the right has at most $(k-1)$ nonzero parts and is in W

since $(h - \mu_1(h) \cdot 1) \cdot w \in W$, contradicting the minimality of k. $\square$

Prop. $\forall \lambda \in H^*$, we have

(1). $M(\lambda)$ has basis $\beta = \{ \prod_{\alpha \in \Phi^+} f_\alpha^{m_\alpha} \cdot v_+ : m_\alpha \geq 0 \}$. (✓. PBW)

(2). $M(\lambda)$ is a hw module w/ h.w. λ. (✓. Page 2.)

(3). $M(\lambda)$ has a unique maximal (proper) submodule $\mathrm{Rad}\, M(\lambda)$

and hence a unique (nontrivial) irr. quotient

$$L(\lambda) := M(\lambda) / \mathrm{Rad}\, M(\lambda).$$

(✓. Previous Page.)

(4) (Universality). If M is any h.w. module of L w/ h.w λ and h.w. vector w^+ (unique up to scaling), then there is a unique module hom. $M(\lambda) \to M$: it must send v^+ to w^+

Pf 1. Construction/existence:

$$\prod_{\alpha \in \Phi^+}^{\rightarrow} f_\alpha^{m_\alpha} v^+ \longmapsto \prod_{\alpha \in \Phi^+}^{\rightarrow} f_\alpha^{m_\alpha} w^+ . \quad (*)$$

uniqueness: $(*)$ is forced once we map v^+ to w^+.

(necessary since homs should preserve wts.)

Pf 2. Use Frobenius Reciprocity,

$$\mathrm{Hom}(M(\lambda), M) = \mathrm{Hom}_{U(L)}\left(\mathrm{Ind}_{U(B)}^{U(L)} \mathbb{C}_\lambda, M\right) \overset{F.R.}{=\!=} \mathrm{Hom}_{U(B)}\left(\mathbb{C}_\lambda, \mathrm{Res}_{U(B)}^{U(L)} M\right) \to$$

1d. spanned by $(1_\lambda \mapsto w^+)$.

1d

(5). If L is any irr. h.w module of $\mathcal{L}$ with h.w. λ, then

$L \cong L(\lambda)$.

Pf: By (4), we have a hom $\theta : M(\lambda) \longrightarrow L$ w/ $v^+ \mapsto w^+$.

It's surj since w^+ generates L, so $L \cong M(\lambda)/\ker\theta$.

L irr $\Rightarrow$ $\ker\theta$ is maximal in L by the corr. thm,

$\Rightarrow \ker\theta = \operatorname{Rad} M(\lambda) \Rightarrow L \cong M(\lambda)/\operatorname{Rad} M(\lambda) = L(\lambda)$. $\square$

Note: The above prop works for any $\lambda \in H^*$.

Corollary. (1). By (5), for any $\lambda \in H^*$ there's a unique h.w. module $L(\lambda)$ up to iso of h.w. λ.

In particular, this is true if $\lambda \in P^+$, so the map $\mathcal{H}w$ is inj.

(2). To prove that $\mathcal{H}w$ is surj, it suffices to show that

$$L(\lambda) = M(\lambda)/\mathrm{Rad}\, M(\lambda) \text{ is f.d. if } \lambda \in P^+.$$

(we already know the "only if")

We'll skip the proof (see Humphreys).

Examples of the bijection $\left\{\begin{matrix}\text{f.d. irreps}\\ \text{of } L\end{matrix}\right\}/\cong \longrightarrow P^+$ in type A.

$$P^+ = \bigoplus_{i=1}^{\ell} \mathbb{N}\underbrace{w_i}_{\text{fund. wts.}}$$

$\langle w_j, \alpha_i \rangle = \delta_{ij}$

A_1 : sl_2.

$\Delta = \{\underbrace{\varepsilon_1 - \varepsilon_2}_{\text{coordinate functions}}\} = \{\alpha_1\}$. $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- fund. wts. $\langle \alpha_1, \alpha_1 \rangle = 2$,

  so setting $w_1 = \frac{1}{2}\alpha_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) = \frac{1}{2}(\varepsilon_1 + \varepsilon_1) = \varepsilon_1$ works.

- $P^+ = \mathbb{N} w_1 \overset{\text{identify}}{=\!=} \mathbb{N}$ ← matches with our indexing of f.d. sl_2-irreps by $\mathbb{N}$ earlier. $\dim V_d = d+1$, $d \geq 0$.

  $n w_1 \longmapsto n$

- Examples:
  - V_2 = the adjoint rep $\langle f \rangle \underset{f}{\overset{e}{\rightleftarrows}} \langle h \rangle \underset{f}{\overset{e}{\rightleftarrows}} \underline{\langle e \rangle} \xrightarrow{e} 0$

    h.w. $h.e = 2e$

    $\lambda(h) = 2 = 2\varepsilon_1(h)$

    $= \underline{\underline{2w_1}}(h)$

  - V_3 $\quad V_{-3} \underset{f}{\overset{e}{\rightleftarrows}} V_{-1} \underset{f}{\overset{e}{\rightleftarrows}} V_1 \underset{f}{\overset{e}{\rightleftarrows}} V_3 \xrightarrow{e} 0$

    $h \cdot v_3 = \lambda(h) v_3$

    $\lambda(h) = 3 = 3\varepsilon_1(h) = \underline{\underline{3w_1}}(h)$

Note: If L is simple, then the adjoint rep $V = L$ (V_2 in the example) is irreducible, so, being a h.w. module, L has a unique h.w.

Weights of the adj. rep. are exactly roots. So we've just deduced that every irr. root system has a unique maximal root. (This is proved as Lemma 10.4.(a) in Humphreys in a different and quite non-trivial way!)

A_{n-1} : $\mathfrak{sl}_n$ $(n \geq 2)$.

$\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots, \varepsilon_{n-1} - \varepsilon_n\}$

- $\alpha_i^\vee = \alpha_i$, so $\langle w, \alpha_i \rangle = (w, \alpha_i)$ $\forall i$.

$\Downarrow$

easy to check: $w_1 = \varepsilon_1$, $w_2 = \varepsilon_1 + \varepsilon_2$, $\cdots$, $w_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1}$

suffices as the fundamental wts.

- dominance order: note that $\varepsilon_1 \succeq \varepsilon_2 \succeq \varepsilon_3 \cdots \succeq \varepsilon_n$.

- Example: the natural module $V = \mathbb{C}^n \circlearrowleft \mathfrak{sl}_n$. The standard basis $\{u_1, \dots, u_n\}$ of $\mathbb{C}^n$ is a wt basis: $\forall h \in H$, say $h = \mathrm{diag}(a_1, \dots, a_n)$,

$$h \cdot u_j = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow j\text{th} = a_j u_j = \varepsilon_j(h) u_j \quad \forall j.$$

So $V = \bigoplus_{j=1}^{n} V_j$, $V_j = \langle u_j \rangle$.

ε_1 is the h.w. so $V \cong L(w_1)$

· **Fact**: $\forall\ 2 \le j \le n-1$, the v.s. $\Lambda^j V$ is an $\mathfrak{sl}_n$-module and is irr. Moreover, $\Lambda^j V$ has highest wt ω_j, i.e.,

$$\Lambda^j V \cong L(\omega_j).$$

Pf: E.X. / H.W.

E.g. for main idea: $j=2$, $n=4$. Start with a "crystal" graph of basis elts of $\Lambda^j V$:

h.w.

$$\begin{tikzcd}
\boxed{u_1\wedge u_2} \arrow[r, shift left, "f_2"] & \boxed{u_1\wedge u_3} \arrow[l, shift left, "e_2"] \arrow[r, shift left, "f_3"] \arrow[d, shift left, "f_1"] & \boxed{u_1\wedge u_4} \arrow[l, shift left, "e_3"] \arrow[d, shift left, "f_1"] \\
& \boxed{u_2\wedge u_3} \arrow[u, shift left, "e_1"] \arrow[r, shift left, "f_3"] & \boxed{u_2\wedge u_4} \arrow[l, shift left, "e_3"] \arrow[u, shift left, "e_1"] \arrow[d, shift left, "f_2"] \\
& & \boxed{u_3\wedge u_4} \arrow[u, shift left, "e_2"]
\end{tikzcd}$$

deduce irreducibility, find max. weight.

Verma modules for sl_2.

- Recall that we identify each $\lambda \in H^*$ with the scalar $\lambda(h)$, so $H^* \leftrightarrow \mathbb{C}$.

  Write $M(\lambda) = U(L) \otimes_{U(B)} \mathbb{C}_\lambda$ as M_c if $\lambda(h) = c$.

$$h \cdot 1_\lambda = \lambda(h) \cdot 1_\lambda \longrightarrow h \cdot 1_\lambda = c \cdot 1_\lambda.$$

- wt. basis: $(*)$

$$0 \xleftarrow{e} \underset{h\cdot = c}{v_+ := 1 \otimes 1_\lambda = 1 \otimes 1_c} \underset{f}{\overset{e}{\rightleftarrows}} \underset{h = c-2}{f v_+} \underset{f}{\overset{e}{\rightleftarrows}} f^2 v_+ \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{h = c-2i}{f^i v_+} \overset{e}{\rightleftarrows} \underset{h = c-2i-2}{f^{i+1} v_+} \cdots$$

- **Ex.:** $\forall\, i \geq 0 \quad e f^{i+1} v_+ = (i+1)(c-i) f^i v_+$.

- $(i+1)(c-i) = 0$ for $i \geq 0$ only if $c = \lambda(h)$ is an integer and equals i.

- If $c = \lambda(h) \in \mathbb{N}$, say $c = i$, we have (by $*$)

describes all Verma modules for $\mathfrak{sl}_2$

$$M_c = M_i = \begin{array}{|c|} \hline v_+ \\ f v_+ \\ \vdots \\ f^i v_+ \\ \hline f^{i+1} v_+ \\ f^{i+2} v_+ \\ \vdots \\ \vdots \\ \\ \hline \end{array}$$

② projs to a basis of $L_c = M_c / \mathrm{Rad}\, M_c$.
$\quad \| $
$\quad L_i$

① Span $\mathrm{Rad}\, M_c$, the unique maximal submodule of M_c, which is inf. dim and iso to M_{-i-2} itself $(c-2i-2 = -c-2)$.

- If $c = \lambda(h) \notin \mathbb{N}$, then $\mathrm{Rad}\, M_c = 0$ and M_c is itself irreducible.
(but it's inf. dim.!)

□

Some skipped proofs / further directions / references / acknowledgements

- Youtube: { Uppsala Algebra / Lie algebras by Walter Mazorchuk (Uppsala U.)
  Jon Brundan / Introduction to Lie Theory (U. Oregon)

  I also based some of my lectures on his old course notes!

- Pfs of Weyl's Complete Reducibility Thm and Serre's Thm. (See those Youtube channels)

- Kac–Moody algebras (Lie algebras defined by gens. and rels. from generalized Cartan data)

- Quantized enveloping algebras $U_q(L) \longrightarrow$ Hopf algebras deforming $U(L)$

- Character theory for reps of s.s. Lie algebras (e.g. Weyl's char. formula. "$\dim L(\lambda)_\mu$")

- Highest weight theory, the Bernstein–Gelfand–Gelfand category $\mathcal{O}$. ...

Thank you !!!