**Last time:** Let $L$ be a s.s. Lie algebra / $k=\bar{k}$, char $k=0$.

**Thm:** There's a bijection

$$HW: \left\{ \begin{matrix} \text{f.d. irreps} \\ \text{of } L \end{matrix} \right\} \Big/ \cong \longrightarrow P^+ = \{\lambda \in H^* : \lambda(h_i) = \langle \lambda, \alpha_i \rangle \in \mathbb{N} \ \forall 1 \le i \le \ell\}.$$

$$(\Delta = \{\alpha_1, \dots, \alpha_\ell\})$$

$$V = \oplus V_\lambda \longmapsto \text{the unique maximal wt } \lambda \text{ in } V \text{ w.r.t. } \le.$$

**Q1.** Why the "$\cong$"?

**A1:** linear algebra: "homomorphisms [inj] preserve wts" $\varphi: V \to W$, say $V_\lambda \ne 0$. $0 \ne v \in V_\lambda$.

Then $\forall h \in H$, $h \cdot \varphi(v) = \varphi(h \cdot v) = \varphi(\lambda(h) v) = \lambda(h) \cdot \varphi(v) \Rightarrow W_\lambda \ne 0$, $\varphi(v) \in V_\lambda \dots$

**Q2.** Why is the maximal wt in $V$ unique?

**A2:** Consequences of having h.w. vectors ...

**Q3:** Why is the map $\mathcal{V} \mathcal{A} \mathcal{W}$ inj. and surj.?

**A3:** h.w. theory (including conseq. of PBW basis theorem) + "Verma modules"

main topic today. "h.w. are nice".

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**Recall:** a highest wt vector in $L \curvearrowright V$ is a vector $v_+$ s.t.

(i) $v_+$ is a wt vector, say $h \cdot v_+ = \lambda(h) v_+$ $\forall h \in H$.

(ii) $e_\alpha \cdot v_+ = 0$ $\forall \alpha \in \Phi^+$ (true if $\lambda$ is maximal)

A h.w. module is one generated by a h.w. vector.

**Prp:** Let $V$ be an irr. $L$-module and let $v_+ \in V$ be a h.v. vector. Then
Say $v_+ \in V_\lambda$.

(1). $v_+$ generates $V$. So $V$ is a h.w. module.

**Pf:** $V$ is irr.

(2). $V \overset{(1)}{=} U(L)\cdot v_+ = \operatorname{Span}\Big\{\underbrace{\overrightarrow{\prod_{\alpha\in\Phi^+}} f_\alpha \cdot \overrightarrow{\prod_{i=1}^{\ell}} h_i \cdot \overrightarrow{\prod_{\alpha\in\Phi^+}} e_\alpha}_{\text{PBW basis once we make } f_\beta < h_i < e_\alpha\ \forall \alpha,\beta\in\Phi^+}\Big\}\cdot v_+ = \operatorname{Span}\Big\{\overrightarrow{\prod_{\alpha\in\Phi^+}} f_\alpha\Big\}\cdot v_+.$

**Pf:** Immediate from the PBW basis theorem and (i), (ii).

(3) $\underbrace{f_{\alpha_{i_1}}^{m_1} f_{\alpha_{i_2}}^{m_2} \cdots f_{\alpha_{i_k}}^{m_k}}_{\in\,\Phi^+} \cdot v_+ \in V_{\lambda - \sum_{j=1}^{k} m_j \alpha_{i_j}}$. Consequently, all wts in $V$ are lower than $\lambda$, so that $\lambda$ is the unique h.w. Moreover, $\dim V_\lambda = 1$,

and more generally $\forall \mu \in H^*$ we have $\dim V_\mu < \infty$ (so $V$ is locally f.d.).

Pf: obvious from (2): consider the wt lattice.

( So we've answered the uniqueness question Q2.)

(5). $\forall\ 1 \le i \le l$, $\langle \lambda, \alpha_i \rangle \ge 0$, so $\lambda \in P^+$.

$\mathbb{N}$, not just $\mathbb{Z}$.

Pf: Consider the restriction $sl_{\alpha_i} \hookrightarrow L \circlearrowright V$ making $V$ an $sl_{\alpha_i}$-module.

By (ii), $e_{\alpha_i} \cdot v_+ = 0$, so $v_+$ is a h.w. vector for $sl_{\alpha_i}$.

It follows from $sl_2$-rep theory (including Weyl's Complete reducibility thm)

$\lambda(h_i) = \langle \lambda, \alpha_i \rangle \in \mathbb{N}$. $\square$ → Done with the "well-definedness" of HW.

## Proving that $\mathcal{A}_W$ is a bijection:

( need the notion of Verma modules )

Recall the triangular decomposition:

$$L = \underbrace{N^+}_{\bigoplus_{\alpha \in \Phi^+} L_\alpha} \oplus H \oplus \underbrace{N^-}_{\bigoplus_{\alpha \in \Phi^-} L_\alpha}$$

Define the positive Borel subalgebra to be $B = N^+ \oplus H$.

**Note:** $N^+, N^-$ are each max. nilp. $B$ is Borel in the sense that it's maximal solvable.

## Definition of Verma modules.

Let $\lambda \in H^*$. (do not need $\lambda \in P$ yet)

- start by letting $\mathbb{C}_\lambda$ be the 1-dim $H$-module $\mathbb{C} = \mathbb{C}\langle 1_\lambda \rangle$

  notation. $1_\lambda$ is just 1

  s.t. $h \cdot 1_\lambda = \lambda(h) \cdot 1_\lambda$.

- turn $\mathbb{C}_\lambda$ into a $B$-module via the Lie algebra hom $B \twoheadrightarrow H$, so that $h \cdot 1_\lambda = \lambda(h) \cdot 1_\lambda \ \forall h \in H$ and $e \cdot 1_\lambda = 0 \ \forall e \in N^+$.

can also define the $B$-actions on $\mathbb{C}_\lambda$ direct this way and checking it makes $\mathbb{C}_\lambda$ a $B$-module.

- Since $\mathbb{C}_\lambda$ is a $B$-module, it's $U(B)$-module. Now define the Verma module of wt $\lambda$ for $L$ to be

$U(L)$-module $M(\lambda)$ defined by

$$M(\lambda) = {}_{U(L)}U(L)_{U(B)} \otimes_{U(B)} {}_{U(B)}\mathbb{C}_\lambda .$$

(So $M(\lambda)$ is the induction of $\mathbb{C}_\lambda$ along the inclusion $U(B) \hookrightarrow U(L)$. and

$$\underset{U(L)}{\underset{\cap}{x}} \cdot (\underset{U(L)}{\underset{\cap}{y}} \otimes 1_\lambda) = xy \otimes 1_\lambda \overset{\text{if } y \in U(B)}{=\!=} x \otimes y 1_\lambda = \begin{cases} \lambda(y)\, x \otimes 1_\lambda & \text{if } y \in H \\ 0 & \text{if } y \in N^+ \end{cases}$$

$$\overset{\text{if } xy \in U(B)}{\|} \quad 1 \otimes xy \cdot 1_\lambda = \begin{cases} \lambda(xy) \otimes 1_\lambda & \text{if } xy \in H, \\ 0 & \text{if } xy \in N^+ \end{cases}$$
.)

**Next time:**

- $M(\lambda)$ is h.w.

- HW is a bijection.