Last time: · Hopf algebra axioms and their consequences for module categories of the algebras.

Today. Start the final topic. f.d. irreps of s.s. Lie algebras

Notation: k : alg. closed field of char 0.

L : s.s. Lie algebra / k

H : a CSA of A

$\Bigg\} \to L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$

Φ root system.

Δ : a fixed base of the root system Φ

assume Δ = $\{\alpha_1, \ldots, \alpha_\ell\}$

$$\longrightarrow \quad L = \underbrace{\bigoplus_{\alpha \in \Phi^+} L_\alpha}_{N^+} \oplus H \oplus \underbrace{\bigoplus_{\alpha \in \Phi^-} L_\alpha}_{N^-}$$

"triangular decomposition"

N^+, N^- are nilp by Engel's Thm.

In fact they are maximal nilp.

W: the Weyl gp

κ: the Killing form on L.

$\to$ $\kappa|_H$ is nondegenerate $\Rightarrow$ $H^* \cong H$, $\alpha \mapsto$ the unique $t_\alpha \in H$ s.t. $\alpha(h) = \kappa(t_\alpha, h)$ $\forall h \in H$.

$(\ ,\ )$ on H^*: $\forall \alpha, \beta \in H^*$, $(\alpha, \beta) \overset{\text{def}}{=} \kappa(t_\alpha, t_\beta)$.

Goal: classify f.d. irreps. of L.

Key ingredients: h.w. considerations / PBW basis thm / linear algebra.

Recall: — Weyl's Thm: Every f.d. rep. of L is a direct sum of simples.

- Hum. Thm 6.4 / E.W. Thm 9.16. "Preservation of Jordan Decomposition"

Let $x \in L$. If $x = s + n$ is the A.J.D of x, then for any rep $\rho : L \to gl(V)$ of L, the J.D. of $\rho(x)$ is

$$\rho(x) = \rho(s) + \rho(n).$$

- $\forall \alpha \in \Phi$, $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. So

$\forall \alpha \in \Phi, \beta \in H^*$, $\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \kappa\left(t_\beta, \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right)$

$$= \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(\alpha, \alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

$$= (\beta, \alpha^\vee) = \langle \beta, \alpha \rangle.$$

- Also recall we can define a relation $\lesssim$ on Φ by declaring $\alpha \leq \beta$ if $\beta - \alpha$ is a $\underline{\underline{\mathbb{N}}}$ - linear comb. of positive (simple) roots.

for us. $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

We may extend $\leq$ to a partial order on H^* by declaring

$$\alpha \leq \beta \text{ for } \alpha, \beta \in H^* \text{ if } \beta - \alpha \in \sum_{\alpha \in \Phi^+} \mathbb{N}\alpha .$$

We call the new order <u>the dominance order</u>.

We can now start the classification ...

First recall that by the result on J.D., any f.d. rep of L

is a direct sum of wt spaces:

$$(*) \qquad V = \bigoplus_{\lambda \in H^*} V_\lambda$$

where $V_\lambda = \{ v \in V \mid h.v = \lambda(h) v \ \forall h \in H \}$. ($\lambda$ is a <u>weight</u> if $V_\lambda \neq 0$, in which case V_λ is a <u>weight space</u>.)

Prop: Let $\alpha \in \Phi$. Then for the above decomp $(*)$,

(1). $L_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$

(2). (integrality) $\qquad \langle \lambda, \alpha \rangle = (\lambda, \alpha^\vee) \in \mathbb{Z}$.

Pf: (1) $\forall v \in V_\lambda$, $x \in L_\alpha$, $h \cdot (\underline{x \cdot v}) = x \cdot h \cdot v + [h,x] \cdot v = x \cdot \lambda(h) v + \alpha(h) x \cdot v = \left(\underset{\substack{\| \\ (\lambda+\alpha)(h).}}{\lambda(h)+\alpha(h)} \right) \underline{x \cdot v}$ ✓

(2). Consider the sl_2-subalgebra in L. V is naturally a sl_α-module via the map $\underbrace{sl_2 \hookrightarrow L \longrightarrow}_{\text{restriction to } sl_2} gl(V)$. So V is a direct sum of simple sl_α-modules. but in such simple modules all wts are integral by sl_2-rep theory. $\square$

Consequences of (1) and (2).

(2) motivates the def. of weight lattices.

<u>Def:</u> We define the <u>weight lattice</u> of (L, H) to be the set

$$P = \{ \lambda \in H^* \mid \lambda(h_i) = \langle \lambda, \alpha_i \rangle \in \mathbb{Z} \ \forall 1 \le i \le \ell \}.$$

(so (2) implies that each wt in $V = \oplus V_\lambda$ must be in P.)

Rmk: Find a dual basis $\beta = \{w_1, w_2, \dots, w_\ell\}$ of Δ in H^*.

i.e. β is the set where $\langle w_j, \alpha_i \rangle = \delta_{ij}$ $\forall\ 1 \le i, j \le \ell$.

Then $w \in P \iff w = \sum_{i=1}^{\ell} a_i w_i$ where $a_i \in \mathbb{Z}$ $\forall\ 1 \le i \le \ell$

$\Downarrow$

$(\langle w, \alpha_i \rangle = a_i \ \forall i)$

So P is just the $\mathbb{Z}$-span of β: $P = \bigoplus_{i=1}^{\ell} \mathbb{Z} w_i$. This is why we call P a lattice.

It will be important to also consider $P^+ := \{\lambda \in H^* \mid \langle \lambda, \alpha_i \rangle \in \mathbb{N}\ \forall i\}$

$= \bigoplus_{i=1}^{\ell} \mathbb{N} w_i$

Def: The elts in β are called <u>fundamental weights</u>.

The wts in P are called <u>integral weights</u>.

P^+ is called the <u>positive weight lattice</u>; its elts are called <u>dominant integral weights</u>

(1) leads to highest weight theory.

Pick a wt μ in $V = \bigoplus V_\lambda$ (so $V_\mu \neq 0$) with μ maximal w.r.t. the dominance order.

then $\forall \alpha \in \Phi^+$, $\underset{\substack{\downarrow \\ \text{(nonzero in } L_\alpha)}}{e_\alpha} \cdot V_\mu \in \underbrace{V_{\mu+\alpha_i} = 0}_{\substack{\downarrow \\ \text{by max. of } \mu}}$. i.e. $e_\alpha \cdot V_\mu = 0 \quad \forall \alpha \in \Phi^+$.

Thus, $\forall\, 0 \neq v_+ \in V_\mu$, we have

$$(*) \begin{cases} e_\alpha \cdot v_+ = 0 & \forall \alpha \in \Phi^+ \\ v_+ \text{ is a wt vector} & \text{(since it's in a wt space).} \end{cases}$$

Def: In a rep of L, a nonzero elt v_+ satisfying $(*)$ is called a highest weight vector. A module generated by a h.w. vector is called a highest weight module.

Rmk: By the above discussion, any f.d. irrep has a h.w. vector and is a highest wt module (since V^+ generates V now that V is irreducible).

It will turn out that the maximal wt. μ must be the unique maximal wt and must be in P^+.

$V \longmapsto$ the unique max wt in V

So we get a map $\{\text{f.d. irreps of } L\} \cancel{/\cong} \longrightarrow P^+$.

$\cong$ → general theory: iso. reps have the same wts.

We'll show that this is a bijection, thus classifying the f.d. irreps via P^+.