Last time: - Definitions of bialgebras and Hopf algebras

- $U(L)$ is a Hopf algebra for a Lie algebra L

A bit more on Hopf algebras.

Recall that a Hopf algebra is a tuple $(A, \mu, \eta, \Delta, \varepsilon, S)$

where

- (A, μ, η) is an algebra — familiar
- (A, Δ, ε) is a coalgebra — flip the diagrams
- Δ, ε are alg. hom — familiar
- S is an antipode (unique and anti-hom if existant)

$$
\begin{tikzcd}
 & A\otimes A \arrow[r, "①"] & A\otimes A \arrow[dr, "\mu"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[r, "\varepsilon"] & k \arrow[r, "\eta"] & A & A \\
 & A\otimes A \arrow[r, "②"] & A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

$① = S\otimes id_A$

$② = id_A \otimes S$

Roughly speaking, each condition in the definition of / each property of a Hopf algebra/bi-algebra A implies some nice property of the category of A-modules:

(1) The comult. allows us to take tensors ("inner") of A modules

enough to have a bialgebra

$$A \to A \otimes A \to \mathrm{End}(V \otimes W) \quad \text{for } V, W \in A\text{-mod}$$

$$a.(v \otimes w) = \Delta(a) \cdot (v \otimes w) = \sum a_1 v \otimes a_2 w. \qquad \text{"}\in Ob\text{"}.$$

(2) The coasso. of Δ implies that the above tensoring is associative:

$\forall$ A-modules X, U, V, the map $\varphi: (X \otimes U) \otimes V \to X \otimes (U \otimes V)$

$$(x \otimes u) \otimes v \longmapsto x \otimes (u \otimes v)$$

is an A-module iso thanks to the coasso. axiom.

<u>pf (sketch)</u>: The key is to show φ is a hom of A-modules.

Let $x \in X$, $u \in U$, $v \in V$. $a \in A$. → in addition to asso. of tensors in A-mod, the bialgebra axioms actually imply that A-mod is a monoidal cat. for any bialgebra A.

$$\varphi(a.((x\otimes u)\otimes v)) = \varphi\left(\sum a_1(x\otimes u)\otimes a_2 v\right)$$

$$= \varphi\left(\sum\left(\sum a_{11}x\otimes a_{12}u\right)\otimes a_2 v\right)$$

$$\left[\left((\Delta\otimes id_A)\circ\Delta\right)(a)\right].(x\otimes u\otimes v) = \sum\sum a_{11}x\otimes(a_{12}u\otimes a_2 v)$$

& "make" k an A-mod.

$$\begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id_A"'] \\ A\otimes A \arrow[u, "id_A\otimes\Delta"] \arrow[r, "\Delta"'] \arrow[ur, phantom, "\circ"] & A \arrow[u, "\Delta"'] \end{tikzcd}$$

$$a\cdot\varphi((x\otimes u)\otimes v) = a.(x\otimes(u\otimes v))$$

$$\left[\left((id_A\otimes\Delta)\circ\Delta\right)(a)\right].(x\otimes u\otimes v) = \sum a_1 x\otimes a_2(u\otimes v)$$

$$= \sum a_1 x\otimes\left(\sum a_{21}u\otimes a_{22}v\right)$$

$$= \sum\sum a_1 x\otimes a_{21}u\otimes a_{22}v$$

□

(3) If A a Hopf algebra, then the fact that the antipode S is an algebra anti-hom implies that for any A-module V,

V^* is automatically an A-module via the action

$$a \cdot f \text{ is the map w/ } (a \cdot f)(v) = f(\,S(a) \cdot v\,)$$

for all $f \in V^*$, $a \in A$.

Pf (sketch): The key: $a \cdot b \cdot f = (ab) \cdot f \quad \forall\, a, b \in A,\ f \in V^*$.

$\forall v \in V$, $\quad (ab \cdot f)(v) = f(\,S(ab)\, v\,) = f(\,S(b)\, S(a) \cdot v\,)$

$\quad (a \cdot \underline{b \cdot f})(v) = b \cdot f\,(\,S(a)\, v\,) = f(\,S(b)\, S(a) \cdot v\,)$ ".

$\square$

(4) When A is a Hopf algebra, the antipode axioms further imply that for any A-module V, the maps

$$V^* \otimes V \to k \qquad , \qquad V \otimes V^* \to k$$

$$f \otimes v \mapsto f(v) \qquad\qquad v \otimes f \mapsto f(v)$$

are hom of A-modules and satisfy certain nice properties that imply that A-mod is a rigid monoidal cat.

(have nice duals)

(5) If A is a cocomm. bialgebra, the map $V \otimes W \to W \otimes V$

true for both group algebras and UEAs.

$T: v \otimes w \mapsto w \otimes v$

is an A-mod iso,

(6) If A is a Hopf algebra/bialgebra that is not cocomm. the modules $V \otimes W$ and $W \otimes V$ may still be iso. in an interesting way (but not through the obvious flip T).

"Quasitriangular structure" on a Hopf algebra A

$\Downarrow$

$V \otimes W \cong W \otimes V$ via an interesting braiding → makes A-mod a braided monoidal category.

(quantum) Yang-Baxter Eq. (universal) R-matrices.

"A quantum gp is a non-commutative and non-cocommutative Hopf algebra."

— Drinfeld ?

↓

e.g. quantum enveloping algebras of Lie algebras. (see Kassel.)

e.g. $U(sl_2)$: as algebra, it's given by gen's and rel's

<u>gens</u>: e, f, h

(*) <u>rels</u>: (S1) - (S4) in EW.

You can also describe the coalgebra structure using the gen.

(*) $\Delta(x) = 1 \otimes x + x \otimes 1$. $\varepsilon(x) = 0$ $\forall$ $x \in sl_2$.

and the antipode has (*) $S(x) = -x$. $\forall x \in sl_2$.

e.g. $\Delta(e) = 1 \otimes e + e \otimes K$

$\Delta(f) = K^{-1} \otimes f + f \otimes 1$

where K is a counterpart of h.

q-deforming all the (*)-ed formulas gives the quantum version $U_q(sl_2)$ of $U(sl_2)$.