Last time. · Pf of the PBW basis theorem via the diamond lemma

· Universal property construction of the free Lie algebra on a set

Today: $U(L)$ as a Hopf algebra (Kassel: "Quantum Groups",

Majid: "A Quantum Groups Primer".)

Let k be a field.

Recall that

1) Given k algebras A, B, the v.s. tensor $A \otimes B$ is automatically an algebra w/ mult. $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$.

one way to view this multiplication:

$$m_{AB} : (A \otimes B) \otimes (A \otimes B) = A \otimes B \otimes A \otimes B \xrightarrow{id_A \otimes \tau_{AB} \otimes id_B} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B .$$

$$a_1 \otimes b_1 \otimes a_2 \otimes b_2 \longmapsto a_1 \otimes a_2 \otimes b_1 \otimes b_2$$

(2) Given algebras A, B, an A-module M and a B module N, the v.s. $V \otimes W$ is naturally an $A \otimes B$ module with action

$$(a \otimes b) \cdot (m \otimes n) = a \cdot m \otimes b \cdot n \quad \forall a \in A, b \in B, m \in M, n \in N.$$

This space is often called the <u>outer tensor</u> of M and N and denoted by $M \boxtimes N$.

(3) The data of a module M over an algebra A is equivalent to the data of an alg. hom $A \longrightarrow \mathrm{End}(M)$. In (2), the module $M \boxtimes N$ gives rise to a hom $A \otimes B \longrightarrow \mathrm{End}(M \otimes N)$.

(4). Now consider the case $A = B$. then given A-modules M, N, by (1)–(3) we have $M \boxtimes N$ as an $A \otimes A$ module, so we have a hom $A \otimes A \to \mathrm{End}(M \otimes N)$.

Want: to make $M \otimes N$ an A-module, i.e., a hom $A \xrightarrow{\rho} \mathrm{End}(M \otimes N)$.

have: $M \otimes N$ is a $A \otimes A$-module, i.e., a hom $A \otimes A \xrightarrow{\rho_2} \mathrm{End}(M \otimes N)$.

Note that it suffices to have an algebra hom $A \xrightarrow{\rho_1} A \otimes A$. ($\rho = \rho_1 \circ \rho_2$ works as desired then.)

This motivates the notion of coalgebras and bialgebras.

To define coalgebras, note that an (assoc.) algebra can be defined as a triple (A, μ, η) where A is a v.s and $\underset{\text{(mult.)}}{\mu : A \otimes A \to A}$, $\underset{\text{(unit)}}{\eta : k \to A}$ are linear maps s.t.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}_A"'] \arrow[dr, phantom, "\circ"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

and

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta \otimes \mathrm{id}_A"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id}_A \otimes \eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

(i.e., the $\searrow \swarrow$ maps are isos).

In addition, A is comm. if

$$\begin{tikzcd} A\otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}.$$

Now, reverse the arrows ...

Def: A coalgebra (over k) is a triple (A, Δ, ε) where A is a v.s. and

counit

comultiplication

$\Delta: C \to C\otimes C$ and $\varepsilon: C \to k$ s.t.

$$\begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "id_A\otimes\Delta"'] \\ A\otimes A \arrow[u, "\Delta\otimes id"] \arrow[ur, phantom, "\circ"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k\otimes A & A\otimes A \arrow[l, "\varepsilon\otimes id_A"'] \arrow[r, "id_A\otimes\varepsilon"] & A\otimes k \\ & A \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}.$$

Furthermore, we call A cocommutative if

$$\begin{tikzcd} A\otimes A \arrow[rr, "\tau_{A,A}"] & & A\otimes A \\ & A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

.

<u>Def</u>: A <u>bialgebra</u> is a tuple $(A, \mu, \eta, \Delta, \varepsilon)$ s.t.

(1) (A, μ, η) is an algebra and (A, Δ, ε) is a coalgebra

(2) The comultiplication $\Delta: A \to A\otimes A$ is an algebra hom. i.e., if we write $\Delta(a) = \sum a_1 \otimes a_2 \; \forall a \in A$ (Sweedler notation.), then

$$\Delta(ab) = \Delta(a)\Delta(b) \quad \forall a, b \in A,$$

i.e., $\sum (ab)_1 \otimes (ab)_2 = \left(\sum a_1 \otimes a_2\right)\left(\sum b_1 \otimes b_2\right) = \cdots\cdots$. (Ex: draw a diagram...)

(3) The counit ε is an algebra hom.

Consequence: If A is a bialgebra, then the tensor $M \otimes N$ of two A-modules M, N is naturally an A-module: $a.(m \otimes n) = \Delta(a) \cdot m \otimes n$

$$= \sum a_1 m \otimes a_2 n.$$

$$A \xrightarrow{\Delta} A \otimes A \longrightarrow \mathrm{End}(M \otimes N)$$

Def. If $(A, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, an antipode of A is a linear map $S : A \to A$ s.t.

$$\begin{tikzcd}
 & A \otimes A \arrow[r, "S \otimes id_A"] \arrow[dr, phantom, "\circ"] & A \otimes A \arrow[dr, "\mu"] & \\
A \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & k \arrow[r, "\eta"] \arrow[dr, phantom, "\circ"] & & A \\
 & A \otimes A \arrow[r, "id_A \otimes S"'] & A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

.

Facts. · If an antipode of a bialgebra exists, it's unique and automatically an algebra antihom: $S(ab) = S(b)S(a)$ $\forall a, b \in A$.

Prop. (1) k is a bialgebra itself (with $\Delta(1) = 1 \otimes 1$, $\varepsilon(1) = 1$.)

(2) For any monoid M, the monoid algebra is a bialgebra w/

$\Delta(g) = g \otimes g$ $\varepsilon(g) = 1$ $\forall g \in M$. It is a Hopf algebra

iff M is a gp, i.e., g^{-1} exists $\forall g \in M$; in this case, $S(g) = g^{-1}$.

(3). For any Lie algebra L, the universal enveloping algebra

$U(L)$ is a Hopf algebra w/ $\Delta(x) = 1 \otimes x + x \otimes 1$, $\varepsilon(x) = 0$ $\forall x \in U$

and $S(x) = -x$. $\forall x \in L$, then induce (Δ should be an alg. hom.)

<u>HW</u>: Prove these.

<u>Q</u>: Why all the axioms? Why an antipode?

<u>A</u>: Roughly, each axiom on A guarantees some nice property of A-module.

— next time.