

Last time: — Equivalence of the two versions of the PBW theorem.

↓ Ver. 2.

— Started the proof of the PBW basis theorem via Bergman's Diamond Lemma.



Continued.

Setup: general: want a linear basis for a quotient of the form

$$k\langle X \rangle / \langle W_\alpha - f_\alpha \mid \alpha \in \Lambda \rangle.$$

↙  
free assoc. algebra

↘  
 $f_\alpha$  is a linear comb. of terms  $\langle W_\alpha$ ,  
 $W_\alpha$  a monomial.

$k\langle X \rangle$  has a mult-respecting  
"semigroup order":  $B < B' \Rightarrow AB < AB'$

will do: obtain the desired basis by taking the irreducible monomials.

nice feature: "if every monomial reduces unambiguously,  
then the irr. do form a basis".

Our algebra  $U(L)$  :  $U(L) = T(L) \cong k\langle X \rangle$ ,  $X = \{x_i \mid i \in I\}$   
 a basis of  $L$   
 $\langle xy - yx - [xy] : x, y \in L$   
 the order  $<$  on  $k\langle X \rangle$   $\xrightarrow{w_x}$   $\underbrace{\hspace{10em}}_{\text{(my) assume } x > y}$   $>$   
 $\Rightarrow$  the deg-lex reduced  
 by any fixed total order on  $X$   $f_x = yx + [xy]$

The irr. monomials are then exactly the  $x_b$  where  $b$   
 $\Rightarrow$  a nondecreasing I-sequence, i.e., they form exactly the proposed  
 PBW basis.

the reason there's no ambiguity in reduction:  $\rightarrow$  (so the irr. do form a basis) Jacobi's identity.

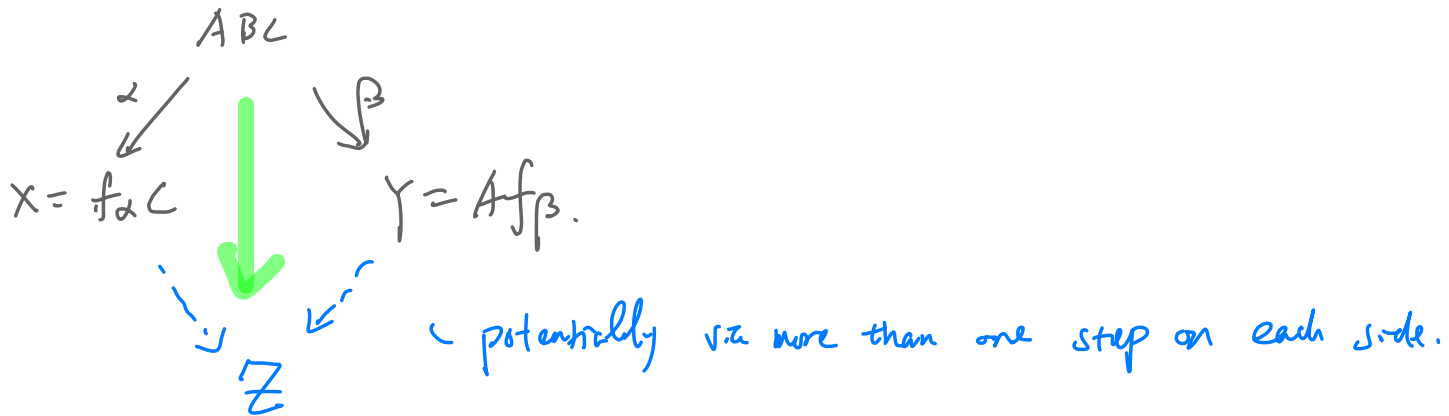
Possible ambiguities from reductions  $AW_2C \rightarrow Af_2C$  :

Note that at least two kinds of ambiguities can occur:

(1) Overlap ambiguity - for reduction of elts of the form

$ABC$

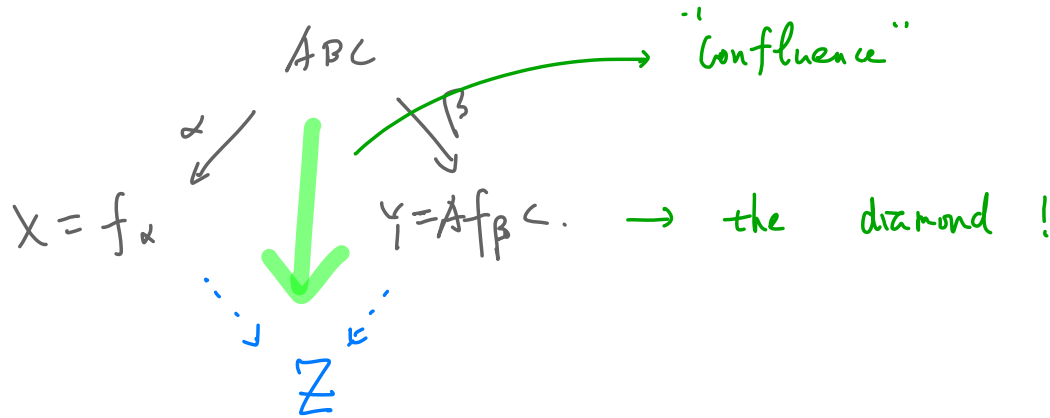
where  $AB = W_\alpha$  and  $BC = W_\beta$  for some  $\alpha, \beta \in \Lambda$ .



(2) Inclusion ambiguity : for reductions of monomials of the form

$$ABC$$

where  $ABC = W_\alpha$  and  $B = W_\beta$  for some  $\alpha, \beta \in \Lambda$ .



In both cases, we say the ambiguity  $\Rightarrow$  resolvable if  $X$  and  $Y$  can be further reduced via the system  $\langle W_\sigma - f_\sigma \mid \sigma \in \Lambda \rangle$  to a common elt.



Thm: (The diamond lemma)

↓  
extremely useful, has rich connections  
to Gröbner-Shirshov basis theory,  
a generalization of Gröbner basis theory  
for commutative algebra.

GS-basis: Coxeter gpi. Hecke algebras,  
Temperley-Lieb algebras

Pf of the PBW basis theorem.

The ideal used is  $\langle x \otimes y - (y \otimes x + [xy]) \mid x, y \in X, x > y \rangle$ .

(It's enough to check these two kinds of ambiguities.) If all overlap and inclusion ambiguities can be resolved, then every monomial in  $k\langle X \rangle$  can be reduced to a unique irreducible elt, and the ir. monomials form a basis for  $k\langle X \rangle / \langle W_2 - f_2 \mid d \in \Lambda \rangle$ . ◻

There's no inclusion ambiguity. The overlap ambiguities occur only on elts of the form  $x \otimes y \otimes z$  where  $x > y > z$ .

We check confluence:

$$x \otimes y \otimes z$$

$$w - w' = [(xy)z] + [(yz)x]$$

+  $[(zx)y] = 0$  by  
the Jacobi identity!

$$(\underline{y \otimes x} + \underline{[xy]}) \otimes z$$

"go smaller"

$$x \otimes (\underline{z \otimes y} + \underline{[yz]})$$

$$\underline{yzx} + \underline{y[xz]} + \underline{[xy]z}$$

$$\underline{zxy} + \underline{[xz]y} + \underline{x[yz]}$$

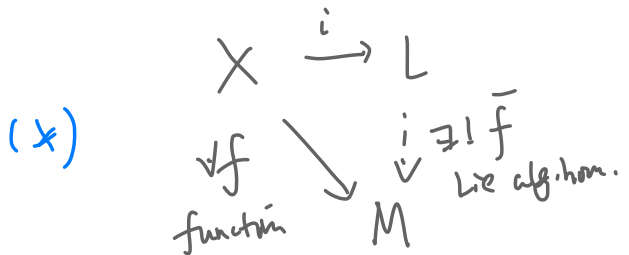
$$W := \underline{zyx} + \underline{[yz]x} + \underline{y[xz]} + \underline{[xy]z} = W' := \underline{zyx} + \underline{z[xy]} + \underline{[xz]y} + \underline{x[yz]}$$

□

# Free Lie algebras (on a set)

Def. Let  $X$  be a set. A free Lie algebra on  $X$  is a Lie algebra  $L$  with a map  $i: X \rightarrow L$  s.t. given any Lie algebra  $M$  and any function  $f: X \rightarrow M$ , there is a unique Lie algebra hom.

$\bar{f}: L \rightarrow M$  st. the diagram

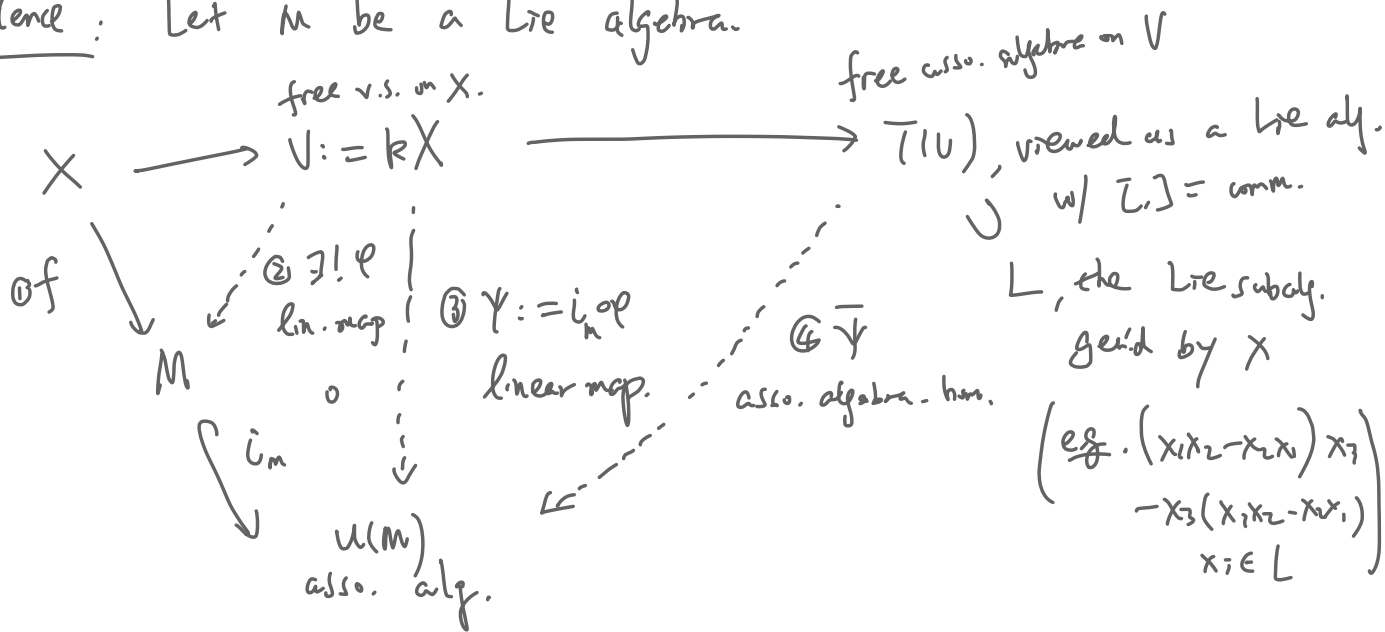


commutes.

Prop: The free Lie algebra on  $X$  exists for any  $X$  and  $\Rightarrow$  unique up to a unique isomorphism.

Pf: Uniqueness: routine as usual.  $\checkmark$

Existence: Let  $M$  be a Lie algebra.



Take  $\bar{f} = \bar{\Psi} \Big|_L$ . ( $\bar{\Psi}: T(U) \rightarrow u(M)$ , want  $\bar{f}: L \rightarrow M$ )

Note that  $\forall x, y \in L$ ,

$$\bar{f}(x \otimes y - y \otimes x) = \bar{f}(x) \bar{f}(y) - \bar{f}(y) \bar{f}(x)$$

$$= [\bar{f}(x), \bar{f}(y)] = [f(x), f(y)] \in M.$$

So  $\bar{f}$  does map inputs to  $M$ , so we may view  $\bar{f}$  as a

map from  $L$  to  $M$ . It clearly makes (\*) commute.  $\square$