Last time: — Equivalence of the two versions of the PBW theorem.

— Started the proof of the PBW basis theorem via Ver. 2.

Bergman's Diamond Lemma.

Continued.

Setup: general: want a linear basis for a quotient of the form

$$k\langle X\rangle \Big/ \langle W_\alpha - f_\alpha \mid \alpha \in \Lambda \rangle .$$

free assoc. algebra

f_α is a linear comb. of terms $< W_\alpha$.

W_α a monomial.

$k\langle X\rangle$ has a mult-respecting "semigroup order": $B < B' \Rightarrow ABC < AB'C$

will do: obtain the desired basis by taking the irreducible monomials.

nice feature: "if every monomial reduces unambiguously, then the irr. do form a basis".

Our algebra $U(L)$: $U(L) = \cancel{T(L)} \cong k\langle X\rangle$, $X = \{x_i \mid i \in I\}$ a basis of L

$\langle \underbrace{x\otimes y}_{w_\alpha} - \underbrace{y\otimes x - [xy]}_{} : x, y \in L \rangle$

the order $<$ on $k\langle X\rangle$ is the deg-lex induced by any fixed total order on X

(may assume) $x > y$

$f_\alpha = y\otimes x + [xy]$

The irr. monomials are then exactly the elts $x_{\underline{b}}$ where $\underline{b}$ is a nondecreasing I-sequence, i.e., they form exactly the proposed PBW basis.

(so the irr. do form a basis)

the reason there's no ambiguity in reduction: Jacobi's identity.

Possible ambiguities from reductions $Aw_\alpha C \longrightarrow Af_\alpha C$:

Note that at least two kinds of ambiguities can occur:

(1) <u>Overlap ambiguity</u>: for reduction of elts of the form

$$ABC$$

where $AB = W_\alpha$ and $BC = W_\beta$ for some $\alpha, \beta \in \Lambda$.

$$\begin{tikzcd}
& ABC \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd] & \\
X = f_\alpha C \arrow[dr, dashed] & & Y = Af_\beta . \arrow[dl, dashed] \\
& Z &
\end{tikzcd}$$

← potentially via more than one step on each side.

(2) **Inclusion ambiguity**: for reductions of monomials of the form

$$ABC$$

where $ABC = W_\alpha$ and $B = W_\beta$ for some $\alpha, \beta \in \Lambda$.

$$\begin{tikzcd}
 & ABC \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd] & \\
X = f_\alpha \arrow[dr, dashed] & & Y = A f_\beta C \arrow[dl, dashed] \\
 & Z &
\end{tikzcd}$$

"Confluence"

→ the diamond!

In both cases, we say the ambiguity is <u>resolvable</u> if X and Y can be further reduced via the system $\langle W_\gamma - f_\gamma \mid \gamma \in \Lambda \rangle$ to a common elt.

Thm: (The diamond lemma)

extremely useful, has rich connections to Gröbner-Shirshov basis theory, a generalization of Gröbner basis theory for commutative algebra.

GS-basis: Coxeter gps. Hecke algebras, Temperley-Lieb algebras

(It's enough to check these two kinds of ambiguities.) If all overlap and inclusion ambiguities can be resolved, then every monomial in $k\langle X\rangle$ can be reduced to a unique irreducible elt, and the irr. monomials form a basis for $k\langle X\rangle / \langle w_\alpha - f_\alpha \mid \alpha \in \Lambda\rangle$. $\square$

Pf of the PBW basis theorem.

The ideal used is $\langle x \otimes y - (y \otimes x + [xy]) \mid x, y \in X, x > y\rangle$.

There's no inclusion ambiguity. The overlap ambiguities occur only on elts of the form $x\otimes y\otimes z$ where $x>y>z$.

We check confluence:

$$
\begin{tikzcd}[column sep=small]
& x\otimes y\otimes z \arrow[dl] \arrow[dr] \arrow[ddd, "\text{“go smaller”}"] & \\
(y\otimes x+[xy])\otimes z \arrow[d] & & x\otimes(z\otimes y+[yz]) \arrow[d] \\
yzx+y[xz]+[xy]z \arrow[d] & & zxy+[xz]y+x[yz] \arrow[d] \\
W := zyx+[yz]x+y[xz]+[xy]z & = & W' := zyx+z[xy]+[xz]y+x[yz]
\end{tikzcd}
$$

$W-W' = [[xy]z]+[[yz]x]+[[zx]y] = 0$ by the Jacobi identity!

□

Free Lie algebras (on a set)

Def. Let X be a set. A free Lie algebra on X is a Lie algebra L with a map $i : X \to L$ s.t. given any Lie algebra M and any function $f : X \to M$, there is a unique Lie algebra hom. $\bar{f} : L \to M$ st. the diagram

$$(*) \qquad \begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] \arrow[d, "f"', "\text{function}"] & L \arrow[d, dashed, "\exists ! \bar{f}", "\text{Lie alg. hom.}"'] \\ & M \end{tikzcd}$$

commutes.

Prop: The free Lie algebra on X exists for any X and is unique up to a unique isomorphism.

Pf: Uniqueness: routine as usual. ✓

Existence: Let M be a Lie algebra.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "① f"'] & \underset{\text{free v.s. on } X.}{V := kX} \arrow[r] \arrow[dl, dashed, "② \exists! \varphi \text{ lin. map}"'] \arrow[dd, dashed, "③ \psi := i_M \circ \varphi \text{ linear map}"] & \underset{\text{free asso. algebra on } V}{T(V)} \arrow[ddl, dashed, "④ \bar{\psi} \text{ asso. algebra hom.}"] \\
M \arrow[dr, "i_M"'] & & \\
& \underset{\text{asso. alg.}}{U(M)} &
\end{tikzcd}$$

$T(V)$, viewed as a Lie alg. w/ $[,] =$ comm.

$\cup$

L, the Lie subalg. gen'd by X

(eg. $(x_1x_2 - x_2x_1)x_3 - x_3(x_1x_2 - x_2x_1)$, $x_i \in L$)

Take $\bar{f} = \bar{\Psi}\big|_L$. $\left(\bar{\Psi} : T(U) \to \mathcal{U}(M), \text{ want } \bar{f} : L \to M\right)$

Notice that $\forall x, y \in L$,

$$\bar{f}\left(x \otimes y - y \otimes x\right) = \bar{f}(x)\bar{f}(y) - \bar{f}(y)\bar{f}(x)$$

$$= [\bar{f}(x), \bar{f}(y)] = [f(x), f(y)] \in M.$$

So $\bar{f}$ does map inputs to M, so we may view $\bar{f}$ as a map from L to M. It clearly makes $(*)$ commute. $\square$