Last time: Two versions of the PBW theorem.

Thm 1. The map ω from

$T = T(L)$, $U = U(L)$

$S = S(L)$

$$\sum_m \left(\begin{tikzcd} T_m \arrow[r] \arrow[rr, dashed, bend right, "\phi_m"'] & U_m \arrow[r, two heads] & (\mathrm{Gr}\,U)_m = U_m/U_{m-1} \end{tikzcd} \right)$$

$$\begin{tikzcd} \overset{\parallel}{\phi}: T \arrow[r] \arrow[d, two heads] & \mathrm{Gr}\,U \\ S \arrow[ur, dashed, "\omega"'] & \end{tikzcd}$$

is an iso.

Thm 2. Fix a basis $\{x_i \mid i \in I\}$ of L and a total order on I.

The (classes of) nondecreasing tensor monomials of basis elts

form a basis of $U(L)$.
$$\left\{ x_{\underline{b}} = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} \;\middle|\; \begin{array}{l} k \geq 0,\ \underline{b} = (i_1, \dots, i_k) \\ i_1 \leq i_2 \leq \cdots \leq i_k \end{array} \right\}$$

Consequences of the theorem:

(a). The map $i : L \to U(L)$ is injective. See last time.

$\hookrightarrow$ deg-1 tensors

(b) Let H be a Lie subalgebra of L. Then the inclusion $H \hookrightarrow L$ induces an inclusion $U(H) \hookrightarrow U(L)$, and $U(L)$ is a free $U(H)$-module.

Pf sketch:

$$\begin{tikzcd}
H \arrow[r, "i_H"] \arrow[d] \arrow[dr] & U(H) \arrow[d, dotted] \\
L \arrow[r, "i_L"] & U(L)
\end{tikzcd}$$

free-module: take a basis $\{x_i \mid i \in I\}$ of H, order I. then extend the basis to a basis $\{x_i \mid i \in I'\}$ $I \subseteq I'$ of L. and extend the order to an order on I' with $x_i < x_j$ $\forall\, i \in I,\ j \in I' \setminus I$. ... $\square$

Equivalence of Thm 1 and Thm 2.

$\mathcal{B} = \{ x_{\underline{b}} \mid \underline{b}$ an nondecreasing I-sequence$\} \subseteq \mathcal{U}$

Last time we observed:

(1) $\mathcal{B}$ is a basis of $\mathcal{S}$, $\mathcal{B}^k$ is a basis of $\mathcal{S}^k$.
(classes of) (classes of)

(2) Spanning is easy thanks to rewriting rules: $\mathcal{B}$ spans $\mathcal{U}$, $\mathcal{B}_k$ spans $\mathcal{U}_k$

(3) to check lin ind / spanning status / basis status in a filtered algebra A (potentially inf-dim) one may pass to the filtered parts A_k (often fin. dim.), Same for inj / surj / iso status of lin. maps between filtered algebras.

Pf that Thm 1 $\Rightarrow$ Thm 2: Assume Thm 1, ie., that w is an iso.

By ②, to check β is a basis it remains to check that β is lin. ind. Take a lin comb $\sum_{\underline{b} \in J} c_{\underline{b}} x_{\underline{b}} = 0$, J a finite set of nondec. I-seq.

We need to show $c_{\underline{b}} = 0 \ \forall \underline{b} \in J$. We do so by induction on the number $m := \{\max \deg \underline{b} : b \in J\}$. Let $J^m = \{\underline{b} \in J \mid \deg \underline{b} = m\}$.

Base case: $m = 0$, $J^m = \{\phi\}$. there's only one summand $c_\phi \cdot x_\phi = c_\phi \cdot 1$. ✓

Induction step: Once we show $c_{\underline{b}} = 0 \ \forall \underline{b} \in J^m$, we could apply induction on $\sum_{b \in J \setminus J^m} c_{\underline{b}} x_{\underline{b}} = 0$ so we are done.

To see $c_{\underline{b}} = 0 \ \forall \underline{b} \in J^m$, consider the diagram

$$\begin{tikzcd}
& \displaystyle\sum_{\underline{b}\in J^m} c_{\underline{b}} x_{\underline{b}} = \sum_{\underline{b}\in J} c_{\underline{b}} x_{\underline{b}} - \sum_{\underline{b}\in J\setminus J^m} c_{\underline{b}} x_{\underline{b}} \arrow[r, mapsto] & 0 - 0 = 0 \\
\displaystyle\sum_{\underline{b}\in J^m} c_{\underline{b}} x_{\underline{b}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \\
& \displaystyle\sum_{\underline{b}\in J^m} c_{\underline{b}} x_{\underline{b}} \arrow[uur, mapsto] &
\end{tikzcd}$$

$$\underbrace{\sum_{\underline{b}\in J} c_{\underline{b}} x_{\underline{b}}}_{\substack{0 \\ \text{by assumption}}} \qquad \underbrace{\sum_{\underline{b}\in J\setminus J^m} c_{\underline{b}} x_{\underline{b}}}_{\deg (m-1) \text{ or lower.}}$$

$$\begin{tikzcd}
& U_m \arrow[dr] & \\
T_m \subseteq T \arrow[ur] \arrow[dr] & & (\mathrm{Gr}\, U)_m \\
& S_m \subseteq S \arrow[ur, "w"] &
\end{tikzcd}$$

But w is an iso, so $\sum_{\underline{b}\in J^m} c_{\underline{b}} x_{\underline{b}} = 0$ and $c_{\underline{b}} = 0 \ \forall\, \underline{b} \in J^m$. $\square$

Pf that Thm 2 $\Rightarrow$ Thm 1: HW. may need something similar to ③ for maps.

Proof of Thm 2. (and hence also Thm 1) "Spanning is easy, lin. ind. is hard."

Humphreys essentially proves lin ind. by constructing a rep. ρ of $\mathcal{U}(L)$ (on $\mathrm{Span}\{x_{\underline{b}}' \mid \underline{b}$ nondecreasing$\}$, formal basis elts, "z_Σ" in 17.4, by mimicking the regular rep) and show that $\rho(\beta)$ form a lin ind. set of operators on the module.

See HW problem on the first Weyl algebra.

We'll prove Thm 2 (the PBW basis theorem) using Bergman's Diamond Lemma.

(google "The main results in this paper are trivial.")

Bergman's Diamond Lemma.

general setup

- start with a set X of letters
- want a basis for a quotient of $R := k\langle X\rangle$, the free algebra on $\langle X\rangle$ by a certain ideal I
- need a "semigroup partial ordering": a partial order $<$

$$B < B' \Rightarrow ABC < AB'C \quad \forall A, B, C \in R$$

setting for $\mathcal{U}(L)$

- $X = \beta = \{x_i \mid i \in I\}$, basis of L.
- $R = k\langle X\rangle$, can be identified w/ $T(L)$.

concat. $\longleftrightarrow$ $\otimes$

ef $\qquad$ $e\otimes f$

- use the deg-lex order for $T(L)$ after putting a total order on I.

- The ideal I is gen. by elts of the form $W_\alpha - f_\alpha$ where W_α is a monomial and f_α is a lin. comb of monomials smaller than W_α in $<$.

So $W_\alpha = f_\alpha$ in R/I and more generally

$$A W_\alpha C \overset{*}{=} A f_\alpha C \quad \forall A, C \in R.$$

We can think of $*$ as a reduction rule.

Thm (roughly) If reductions can be done with no ambiguity, then the minimal/irr. monomials in $\langle X \rangle$ form a basis of R/I.

- $I = \langle x \otimes y - y \otimes x - [x,y] \rangle$.

may assume we only take pairs x, y where $x > y$, so

$$x \otimes y - y \otimes x - [x,y]$$

$$= W_\alpha - f_\alpha$$

w/ $W_\alpha = x \otimes y$, $f_\alpha = \underset{\overset{\wedge}{x \otimes y}}{y \otimes x} - \underset{\overset{\wedge}{x \otimes y}}{[x,y]}$.

to be continued...