Last time: Universal enveloping algebra

- Universal property $\qquad \mathrm{Hom}_{\mathrm{Lie}}(L, \mathrm{Lie}(A)) = \mathrm{Hom}_{\mathrm{Asso.}}(U(L), A)$

- existence and uniqueness

$$
\begin{tikzcd}
 & T(L) \arrow[rr, two heads] \arrow[dd, dashed, "\exists!\varphi_1 \text{ alg. hom}"] & & U(L) := \dfrac{T(L)}{\langle x\otimes y - y\otimes x - [xy] \rangle} \arrow[ddll, dashed, "\exists!\varphi_2 \text{ alg. hom}"] \\
L \arrow[ur] \arrow[dr, "\forall\varphi"'] & \circ & \circ & \\
 & A & &
\end{tikzcd}
$$

$\varphi([xy]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$.

(φ is a Lie alg. hom)

We'll see a similar diagram for free Lie algebras.

Today: The PBW Theorem (Poincaré – Birkov – Witt)

one version will give a basis of $U(L)$.

Preparation. Recall that (again, all algebras are unital below)

- A graded algebra over k is an algebra with a direct sum decomp

$$A = A^0 \oplus A^1 \oplus \cdots \oplus \underbrace{A^k}_{\text{degree } k \text{ (homogeneous) part}} \oplus \cdots$$

as a v.s. s.t. $A^i \cdot A^j \subseteq A^{i+j}$ and $1_A \in A^0$. (e.g. poly alg. $k[x_1, \ldots, x_n] = S(V)$. V a space w/ basis $x_1, \ldots, x_n$.)

- A filtered algebra is an algebra w/ filtration of v.s.

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k \subseteq \cdots$$

s.t. $A = \bigcup_i A_i$ and $A_i \cdot A_j \subseteq A_{i+j}$.

— Clearly, if $A = \bigoplus A^i$ is graded, it's also filtered w/ filtration

$$A_0 \subseteq A_1 \subseteq \cdots$$

where $A_j = \bigoplus_{i \leq j} A^i$. $\left(A = k[x_1, \ldots, x_n] < \begin{array}{l} A^j = \{ p \in A \mid \deg p = j \} \\ A_j = \{ p \in A \mid \deg p \leq j \} \end{array}\right)$

— Conversely, there's a standard way to obtain a graded algebra from any filtered algebra A; this is the <u>associated graded algebra</u> $\operatorname{Gr} A$ defined as follows:

- $\forall\, i \geq 0$, set $(\operatorname{Gr} A)^i = A_i / A_{i-1}$ as a v.s.
- take $\operatorname{Gr} A = \bigoplus_i (\operatorname{Gr} A)^i$.
- to define the bilinear mult, extend the multiplication $(\operatorname{Gr} A)^i \times (\operatorname{Gr} A)^j \to \operatorname{Gr} A^{i+j}$

note it's well-defined!

$$\big((a_i + A_{i-1}), (a_j + A_{j-1})\big) \mapsto (a_i a_j + A_{i+j-1})$$

Remark (Informal): To do mult. in $Gr A$ is like doing mult. in A but only considering highest degree terms.

The PBW Theorem. Let L be a Lie algebra. Let $T = T(L)$, $S = S(L)$, $U = U(L)$, and let $\pi_S : T \twoheadrightarrow S$, $\pi_U : T \twoheadrightarrow U$ be the natural projections.

$\forall k \geq 0$, consider the map

$$\phi_k : L^{\otimes k} = T^k \hookrightarrow T_k \twoheadrightarrow U_k := \pi(T_k) \twoheadrightarrow U^k := U_k / U_{k-1}$$
$$= (Gr U)^k$$

Assemble the maps $(\phi_k)_{k \geq 0}$ to a map $\phi = \sum \phi_k$.

(So $\phi(v) = \phi_k(v)$ if $v \in L^{\otimes k}$).

Note that $\forall x, y \in L$, $\phi(x \otimes y - y \otimes x) = \phi_2(x \otimes y - y \otimes x) \overset{U^2}{=} [x, y] + U_1 = 0 \in (Gr U)^2$

therefore ϕ factors through $S(L)$, giving us a map $\omega : S(L) \to \mathrm{Gr}\, U$.

$$\begin{tikzcd} T(L) \arrow[r, two heads] \arrow[dr, "\phi"'] & S(L) \arrow[d, dashed, "\omega"] \\ & \mathrm{Gr}\, U \end{tikzcd}$$

$\phi(x \otimes y - y \otimes x) = 0$
$\forall x, y \in L$

<u>Thm 1</u> (PBW, Version 1, "coordinate-free") ω is an isomorphism.

<u>Thm 2.</u> (PBW, Version 2, "PBW basis thm") Let $\{x_i \mid i \in I\}$ be a basis of L.
Fix a total order $<$ on L. Define an I-tuple to be a tuple $\underline{b} = (i_1, \ldots, i_k)$, $k \in \mathbb{Z}_{\geq 0}$ where $i_j \in I$ $\forall 1 \leq j \leq k$. Call $\underline{b}$ nondecreasing if $i_1 \leq i_2 \leq i_3 \cdots \leq i_k$, and define

$$x_{\underline{b}} := x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} \in U \qquad (\text{take } x_\phi = 1).$$

Then the set $\beta := \{ x_{\underline{b}} \mid \underline{b} \text{ is an non-dereasing } I\text{-tuple} \}$ is a basis of U.

Immediate corollary: The map $i: L \to U(L)$ is inj.
$$x_i \mapsto x_i$$

E.g. If $L = sl_2 = \langle e, f, h \rangle$, if we declare $e < f < h$, then by the theorem a basis of $U(L)$ is $\{ e^i f^j h^k \mid i, j, k \geq 0 \}$.

Equivalence of the two theorems.

the length of $\underline{b}$

Note: (1). Let $\beta^k = \{ x_{\underline{b}} \in \beta : \deg \underline{b} = k. \} \ \forall k \geq 0$. Then (the classes of the elts of) β^k is a basis of $S^k \ \forall k$.

In particular, $\dim S^k = |\beta^k| = \binom{n+k-1}{k}$, $\dim S_k = |\beta_k| = \binom{n+k}{k}$.

(2) It's easy to see that β spans $U(L)$ and β_k spans U_k: since we mod out the relations $x \otimes y = y \otimes x + [x, y]$ to obtain U from T, we can rewrite every monomial to

a linear comb of nondecreasing monomials of smaller or equal deg.

e.g. $\underline{sl_2}$: $e < f < h$. $h\underset{\sim}{fe}h = h(ef-h)h$

$= hefh - h^3$

(Thus, the hard part of Thm 2 is showing β is lin. ind.)

3). To show a set β in a filtered algebra A $\left\{\begin{array}{l}\text{is lin. ind} \\ \text{spans } A \\ \text{is a basis of } A\end{array}\right\}$, it suffices

to show that for each $k \geq 0$, the set $\beta \cap A_k$ $\left\{\begin{array}{l}\text{is lin ind} \\ \text{spans } A_k \\ \text{is a basis of } A_k\end{array}\right\}$. $\to$ useful if

$\dim A = \infty$ but $\dim A_k < \infty$ $\forall k$.

We'll use these ideas to prove the equivalence of the two theorems. $\leftarrow$ next time. try yourself first!