Last time: Serre's Thm. Roughly: A s.s. Lie algebra L (w/ usual data of H, Φ, Δ, say $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, C) is generated by

$$\{ e_i := e_{\alpha_i},\ f_i := f_{\alpha_i},\ h_i = h_{\alpha_i} \mid 1 \le i \le \ell \}$$

subject ~~only~~ to the relations

more on gen. and relations later

(1) $[h_i, h_j] = 0 \quad \forall i,j$

(2) $[h_i, e_j] = \langle \alpha_j, \alpha_i \rangle e_j = c_{ji} e_j$.

$[h_i, f_j] = -c_{ji} f_j$

(3) $[e_i, f_j] = \delta_{ij} h_i$

(4) $(\operatorname{ad} e_i)^{1-c_{ji}}(e_j) = 0$, $(\operatorname{ad} f_i)^{1-c_{ji}} f_j = 0$.

Today: Universal Enveloping Algebras.

Assumption: All algebras are unital. k is an arbitrary field.
↓
for the rest today.

following: Humphreys. Kiyoshi Igusa's notes

people.brandeis.edu/~igusa/Math223F11/Note223a.pdf

I. The universal enveloping algebra of a Lie algebra L

Punchline: It will be an associative algebra $\mathcal{U}$ s.t. modules of $\mathcal{U}$ and modules of L are the same things. (as a gp algebra is to a group.)

Def 1: The <u>universal enveloping algebra (UEA)</u> of a Lie algebra L over k (here L is allowed to be inf. dimensional) is a pair (U, i) where U is an associative algebra and i is a map $i : L \to U$ w/.

$$i([xy]) = \left([i(x), i(y)]_U = \right) i(x)\, i(y) - i(y)\, i(x)$$

(i.e., it's an associative algebra with a Lie algebra hom from L) which satisfies the following universal property: for any associative algebra A and any linear map $\varphi : L \to A$ s.t. $\varphi([xy]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ $\forall x, y \in L$, there exists a unique asso. algebra homomorphism for which the diagram

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "\forall \varphi \text{ lin. resp. } [\,]"'] & U \arrow[d, dashed, "\exists ! \bar{\varphi}"] \\ & A \end{tikzcd}$$

commutes.

Thm: A UEA of a Lie algebra exists, and it's unique up to a unique isomorphism (of associative algebras).

Pf: Existence: we'll construct it explicitly using the universal properties of tensor algebras and quotients. (soon).

Uniqueness (HW): usual "abstract nonsense". $\square$

Why care? Let U be the UEA of a Lie algebra L with $i: L \to U$.

- Recall that an L-rep is a v.s. V with a Lie algebra hom

$$\varphi: L \to \underset{A}{\underline{gl(V)}} \qquad \left(V \text{ is a } L\text{-module w/ action } x \cdot v = \varphi(x)(v) \right)$$

- A U-module is of course a v.s. V with an algebra hom

$$\rho: U \to \underbrace{\mathrm{End}(V)}_{\substack{\| \\ A \text{ as v.s.}}} \qquad \big(V \text{ is a } U\text{-module with action } u\cdot v = \rho(u)(v)\big).$$

The commutative diagram in Def 1 says that any $\underbrace{L\text{-module}}_{\Downarrow}$ V is naturally a U-module: induce ρ as $\bar{\varphi}$ in the above notation.

$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ $\forall x, y \in L$

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "\varphi"'] & U \arrow[d, dashed, "\bar{\varphi} =: \rho"] \\ & A = \mathrm{End}(V) \end{tikzcd}$$

Conversely, any U-module is naturally an L-module: take $\varphi = \rho \circ i$.

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, dashed] & U \arrow[d, "\rho"] \\ & \mathrm{End}(V) = \mathfrak{gl}(V) \end{tikzcd}$$

So, modules of L and of U are the same things.

so one can use tools from (rep) theory of associative algebras/rings.

II. Algebra review

· Tensor algebras

Given any vector space V, the tensor algebra of V is the vector space $T(V) = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus \cdots$

where $V_k = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ copies of } V} = V^{\otimes k}$, with $V_0 = k$. The product on $T(V)$ is tensoring.

Note that V naturally embeds into $T(V)$ by the map $i: V \to V_1 \hookrightarrow T(V)$.

Also recall that $T(V)$ is <u>the universal associative algebra gen. by V</u> in the following sense:

For any associative algebra A and any k-linear map $\varphi: V \to A$,

there exists a unique algebra hom $\bar{\varphi}: T(V) \to A$. s.t. the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "\forall \varphi \text{ lin. map}"'] & T(V) \arrow[dl, dashed, "\exists ! \bar{\varphi} \text{ alg. hom.}"] \\ A & \end{tikzcd}$$

commutes.

Pf: <u>Existence</u>: let $\bar{\varphi}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \overset{(*)}{=} \varphi(v_1)\varphi(v_2)\cdots\varphi(v_k)$, extend linearly and check that it works.

<u>Uniqueness</u>: No choice, $(*)$ is forced by the comm. diagram.

Some related notions for Wednesday.

- The symmetric algebra on V is the algebra S with the universal property

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "\forall \text{lin.} \varphi"'] & S \arrow[d, dashed, "\exists ! \text{ alg. hom } \bar{\varphi}"] \\ & A \text{, any comm. alg} \end{tikzcd}$$

It exists and may be constructed as

$$S(V) := \frac{T(V)}{\langle x\otimes y - y\otimes x \rangle}.$$

Recall that if $\{x_1, x_2, \ldots, x_l\}$ is a basis of V, then

$T(V)$ is naturally iso to the free asso. algebra $k\langle x_1, \ldots x_l \rangle$.

$S(V)$ . . . . — — $\underbrace{\text{polynomial}}_{\text{free comm.}}$ algebra $k[x_1, \ldots, x_l]$.

IV. Construction of the UEA.

Let $U(L) = T(L) \big/ \langle x\otimes y - y\otimes x - [x,y] \rangle$

(So in the quotient $[x,y] = x\otimes y - y\otimes x$.)

Thm: $U(L)$ is a UEA of L $\left(\text{with } i : L \xrightarrow{i_{T(L)}} T(L) \twoheadrightarrow U(L)\right)$.

Pf:

$$
\begin{tikzcd}
L \arrow[r, "i_{T(L)}"] \arrow[dr, bend right, "\varphi"'] & T(L) \arrow[r, two heads] \arrow[d, dashed, "\exists!\,\phi"] & T(L) \big/ \langle [x,y] - x\otimes y + y\otimes x \rangle . \arrow[dl, dashed, "\exists!\,\bar\varphi"] \\
 & A &
\end{tikzcd}
$$

✓

Since $\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(x)\varphi(y)$, $\bar\varphi([x,y] - x\otimes y + y\otimes x) = 0$. □