Last time: Conjugacy Thm: Any two CSA's of a s.s. Lie algebra are conjugate.

Consequence: The map $f: \left\{ \begin{array}{c} \text{s.s.} \\ \text{Lie algebras} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{root} \\ \text{system} \end{array} \right\}$

$$L \overset{H}{\longmapsto} \Phi$$

is well-defined.

To complete our project of classifying complex s.s. Lie algebras,

one thing remains: show that f is bijective.

Surjectivity: Every (irr) root system / (conn.) Dynkin diagram can be realized via a (existence theorem) Cartan decomposition of a s.s. (simple) Lie algebra.

$F_4 \hookrightarrow E_6$. harder.

Method 1: ABCD: seen. E: also relatively easy F_4, G_2: Kac's 'folding' tech.

Injectivity: "Isomorphism Thm": If (L, H), (L', H') lead to isomorphic root systems, then $L \cong L'$.

L, L' s.s., H, H' CSA's

(uniqueness thm)

Two proofs: (a). Hum Thm 14.2.

(b). "a higher powered proof" using "generators and relations" (Serre's Thm).

Today: Serre's Thm, and how it proves both the existence and uniqueness theorems.

will also give a second proof for surj:

for each root system Φ, we can construct a Lie algebra L by generators and relations (using the Dynkin diagram / Cartan data) s.t. L's root system $\cong \Phi$.

Serre's Thm:

Preparation: Prop 1 (E.W. 14.5): L s.s. $H \subseteq L$ CSA $\to \Phi$ root system.

Fix a base $\Delta \subseteq \Phi$. say $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$.

Find $(e_i := e_{\alpha_i}, f_i := f_{\alpha_i}, h_i := h_{\alpha_i})$ for each $1 \le i \le \ell$

from $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$. Then the following relations hold.

(1). $[h_i, h_j] = 0 \quad \forall i, j$. ✓ H is abelian.

(2). $[h_i, e_j] = c_{ji} e_j$ and $[h_i, f_j] = -c_{ji} f_j \quad \forall i, j$ where $c_{ij} = \langle \alpha_i, \alpha_j \rangle$

similar.

(✓. $[h_i, e_j] = \alpha_j(h_i) e_j = \alpha_j\left(\frac{2t_{\alpha_i}}{(\alpha_i, \alpha_i)}\right) e_j = \left(\frac{t_{\alpha_i}, 2t_{\alpha_j}}{(\alpha_i, \alpha_i)}\right) e_j = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)} e_j$

$= c_{ij} e_j$.)

(3). $[e_i, f_i] = h_i \ \forall i$; $[e_i, f_j] = 0 \ \forall i,j$ s.t. $i \neq j$.

✓ obvious.

✓ $[e_i, f_j] \subseteq [L_{\alpha_i}, L_{-\alpha_j}] \subseteq L_{\alpha_i - \alpha_j}$ and $L_{\alpha_i-\alpha_j} = 0$ since $\alpha_i - \alpha_j \notin \Phi$ (because Δ is a base).

(4) (Serre relations) $(\mathrm{ad}\, e_i)^{1-c_{ji}}(e_j) = 0$; $(\mathrm{ad}\, f_i)^{1-c_{ji}}(f_j) = 0$.

similar

Pf: It remains to prove $(\mathrm{ad}\, e_i)^{1-c_{ji}}(e_j) = 0$. To do so, consider the space

$$M := \bigoplus_{\substack{k \in \mathbb{Z} \\ \alpha_j + k\alpha_i \in \Phi}} L_{\underbrace{\alpha_j + k\alpha_i}}.$$

α_i - root string through α_j.

Recall that M is an irreducible $sl(\alpha_i)$-module. ($\to$ "q.r.") Note that:

① no negative int. appears as k since Δ is a base. ② $k = 0$ does appear in the sum. $\to \alpha_j$

So, by the "integrality properties", the largest k appearing in the sum satisfy $0 - k = \langle \alpha_j, \alpha_i \rangle = c_{ji}$. So $k = -c_{ji} > 0$.

Thus, $(\mathrm{ad}\, e_i)^{-c_{ji}}(e_j) \in L_{\alpha_j - c_{ji}\alpha_i}$ is a highest wt vector, so

$$\mathrm{ad}(e_i)\left[(\mathrm{ad}\, e_i)^{-c_{ji}}(e_j)\right] = (\mathrm{ad}\, e_i)^{1-c_{ji}}(e_j) = 0. \quad \square$$

Note: · The proof shows that $1 - c_{ji}$ is the minimal integer k s.t. $(\mathrm{ad}\, e_i)^k(e_j) = 0$.

· As we'll see from Serre's Thm. Relations (1) – (4) is a full set of relations characterizing L.

Serre's Thm: Let C be the Cartan matrix of a root system (with base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$). Let L' be the complex Lie algebra generated by the elts e_i', f_i', h_i' for $1 \le i \le \ell$ subject to the relations (1)–(4) from the previous prop.

Then L' is finite-dimensional and semisimple, the elts $\{h_1', \ldots, h_\ell'\}$ span a CSA H' of L', and the root system of L' has Cartan matrix C.

Pf: See Hum. 18.3. (HW: Read the proof.)

Note: The proof also shows that $\{h_1', \ldots, h_\ell'\}$ is a basis of H'.

Corollary: In Prop 1, (S1)–(S4) is a full set of relations for L. i.e., L is iso to the Lie alg. gen by $\{e_i, f_i, h_i\}$ subject to (S1)–(S4).

Pf: Since L satisfy (S1)-(S4), $\exists$ a Lie algebra hom $\varphi: L' \to L$.

$e_i' \mapsto e_i$, $f_i' \mapsto f_i$, $h_i' \mapsto h_i$. So it suffices to check that

φ is an iso. But φ is surj since L is generated by

H and $\{L_\alpha : \alpha \in \Delta\}$, and $\dim L = \dim H + |\Phi| = \ell + |\Phi|$

↙Note

$= \dim H' + |\Phi|$

$= \dim L'$.

So we are done. $\square$

Thm. (The isomorphism theorem. Hum 18.4. (b), 14.2) Let L, L' be s.s. Lie algebras, with resp CSA's H and H' and root systems Φ and Φ'.

Let an isomorphism $\Phi \to \Phi'$, $\alpha \mapsto \alpha'$ be given.

Then (recall) the iso send a base Δ of Φ to a base Δ' of Φ' and induces an iso. $\pi : H \to H'$. For each $\alpha \in \Delta$, if we select any nonzero elts $x_\alpha \in L_\alpha$ and $x'_{\alpha'} \in L'_{\alpha'}$, then there exists a unique iso $\pi : L \to L'$ extending $\pi : H \to H'$ and such that

$\pi(x_\alpha) = x'_{\alpha'}$ $\forall \alpha \in \Delta$.