Last time: Almost completed studying the corresp.

$$\begin{tikzcd} \left\{\begin{matrix}(\text{simple})\\ \text{s.s. Lie algebras}\end{matrix}\right\} \arrow[r, leftrightarrow, bend left, "\text{nice corresp}"] \arrow[r, leftarrow, shift left] \arrow[r, shift right, "f"'] & \left\{\begin{matrix}(\text{irr})\\ \text{root systems}\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix}(\text{conn.})\\ \text{Dynkin diagrams}\end{matrix}\right\} \end{tikzcd}$$

$$\begin{tikzcd} f: \; L \underbrace{+ \text{ a CSA } H}_{\text{choice}} \arrow[r, mapsto, shift left] & \Phi \arrow[l, mapsto, shift left, "?"] \end{tikzcd}$$

DONE.

$$\begin{tikzcd} \Phi \arrow[r, mapsto, shift left, "\Delta"] & D \arrow[l, mapsto, shift left, "\text{recover}"] \arrow[d, phantom, "\wr"] \\ & \text{Cartan data} \end{tikzcd}$$

Today: A closer look at the left side.

Two questions:

1. why is f independent of the choice of H?

2. why is f a bijection? In particular, how to recover L from Φ?

But first a missing proof.

Thm. Let L be a complex s.s. Lie algebra. Let H be a CSA of L, and let Φ be the corr. root system. Then L is simple $\Leftrightarrow$ Φ is irr.

Pf: $(\Leftarrow)$. Suppose Φ is irr. Consider the Cartan decomp.

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha \qquad (*)$$

Suppose that L is not simple. say L has a proper, nontrivial ideal $I \subseteq L$.

Recall that $L = I \oplus I^\perp$. Since H is toral, each of I and I^2 has an eigenbasis under the action (w.r.t. Killing form) ad H. Since $\dim L_\alpha = 1$ $\forall\, \alpha \in \Phi$,

it follows that $I = H_1 \oplus \bigoplus_{\alpha\in\Phi_1} L_\alpha$ and $I^\perp = H_2 \oplus \bigoplus_{\alpha\in\Phi_2} L_\alpha$ for some

Subspaces H_1, H_2 of H and some subsets Φ_1, Φ_2 of Φ s.t. $\Phi_1 \cap \Phi_2 = \phi$, $\Phi_1 \cup \Phi_2 = \Phi$. We claim that $\Phi_1 \neq 0$, $\Phi_2 \neq 0$ and $\Phi_1 \perp \Phi_2$ (i.e., $(\alpha, \beta) = 0$ $\forall \alpha \in \Phi_1, \beta \in \Phi_2$). This means Φ is reducible, contradiction.

To prove the claims, first note that if $\Phi_2 = \phi$ then $\Phi_1 = \Phi$. Since L is gen. by $\{L_\alpha : \alpha \in \Phi\}$, it follows that $I = L$, contradicting properness of I. So $\Phi_2 \neq 0$. Similarly, $\Phi_1 \neq 0$. Take $\alpha \in \Phi_1$, $\beta \in \Phi_2$. Let $\{e_\alpha, f_\alpha, h_\alpha\}$ and $\{e_\beta, f_\beta, h_\beta\}$ be the $\mathfrak{sl}_2$-triples for α and β. Then $h_\beta = [e_\beta, f_\beta] \in I^\perp$.

So
$$\alpha(h_\beta) e_\alpha = [h_\beta, e_\alpha] \in I \cap I^\perp = 0$$

So $\alpha(h_\beta)e_\alpha = 0$. So $\alpha(h_\beta) = (\alpha, \beta^\vee) = 0$.

This implies $(\alpha, \beta) = 0$ so $\Phi_1 \perp \Phi_2$, as desired.

($\Longrightarrow$). Suppose L is simple. Suppose, for contradiction, that Φ is reducible. say with $\Phi = \Phi_1 \cup \Phi_2$, $\Phi_1 \neq \phi$, $\Phi_2 \neq 0$. $\Phi_1 \cup \Phi_2 = \Phi$. $\Phi_1 \cap \Phi_2 = \phi$. $\Phi_1 \perp \Phi_2$.

For any $\alpha \in \Phi_1$, $\beta \in \Phi_2$,

$$(\alpha+\beta, \beta) = (\alpha, \beta) + (\beta, \beta) = (\beta, \beta) \neq 0 \quad \text{since } \beta \neq 0.$$

Similarly $(\alpha+\beta, \alpha) \neq 0$. So $\alpha+\beta \notin \Phi_1 \cup \Phi_2 = \Phi$, i.e., $\alpha+\beta$ is not a root, so $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} = 0$, i.e. $[L_\alpha, L_\beta] = 0$ $\forall \alpha \in \Phi_1, \beta \in \Phi_2$.

Let L_1 be the subalgebra of L generated by $\{L_\alpha : \alpha \in \Phi_1\}$; define L_2 similarly, Note that:

- If $L_1 = L$, then $[L, L_2] = 0$, so $L_2 \subseteq Z(L)$, contradicting the simplicity of L.

See EW. Prop 14.2 for a more detailed proof.

- So $L_1 \subsetneq L$. But then $[L_1, L] = [L_1, L_1] \subseteq L_1$, so L_1 is a proper, nonzero ideal in L, contradicting the simplicity of L. ◻

Back to the questions on the map $f : (L, H) \longmapsto \Phi_H$.

Q 1. Why is f ind of the choice of H?

Key fact: Fix L (s.s.). any two CSAs H and H' of L are conjugate under $\mathrm{Inn}(L)$ in that $H' = \phi(H)$ for some $\phi \in \mathrm{Inn}(L)$

Conjugacy Thm.

(Recall that $\mathrm{Inn}(L) = \langle \exp \mathrm{ad}\, x : \mathrm{ad}\, x \text{ nilp} \rangle \leq \mathrm{Aut}(L)$.)

see Lecture 3

Remarks:

- The theorem appears as Thm 12.6. and is proved in Append. C. in EW. The proof partially "cheats" by using numerics in the exceptional types (using $\dim L = \dim H + |\Phi|$)

  not too much machinery, but uses classification of root systems

- In Hum., the theorem is treated in Ch 15 & Ch 16., which proves the stronger fact that any two Borel subalgebras of L are (def: max. solvable subalg. see HW) conj under a certain subgp $\mathcal{E}(L)$ of $\mathrm{Inn}(L)$. (involves strongly ad-nilp elts)

more conceptual, heavy machinery, doesn't need root system classification

Why does the conj. of H, H' imply $\Phi_H \cong \Phi_{H'}$?

Answer: Rough reason: Say $H' = \phi(H)$. $\phi \in \mathrm{Inn}(L)$.

Being an automorphism, ϕ preserves $[\,,\,]$, so it should "translate" Cartan decompositions. Moreover, ϕ respects the Killing form. So it preserves all geom. info about the root systems.

$$\alpha'(\overset{\in H'}{\underline{\phi(h)}}) = \alpha(h) \quad \forall h \in H.$$

every elt in H' is of this form

(1) How to find an iso map $\Phi_H \to \Phi_{H'}$.

$H \to L = H \oplus \bigoplus_{\alpha\in\Phi_H} L_\alpha$. Take $h \in H$, $\alpha \in \Phi_H$, $x \in L_\alpha$. Then

$$[\overset{\in H'}{\phi(h)}, \phi(x)] = \phi[h,x] = \phi(\alpha(h)x) = \alpha(h)\,\phi(x).$$

So $\phi(x)$ is in the α'-wt space of H' where $\alpha' \in \Phi_{H'} \subseteq H'^*$ is defined by

Thus we have a map $g: \Phi_H \to \Phi_{H'}$, $\alpha \mapsto \alpha'$.

It's easy to see it's a bijection. (One proof: consider the obvious analog map from $\Phi_{H'} \to \Phi_H$, which will be a two-sided inverse).

② Why g preserves $\langle , \rangle$.

Reason: $\forall a,b \in L$, $K(\phi(a), \phi(b)) = Tr(ad\,\phi(a)\, ad\,\phi(b)) \overset{HW}{=\!=} Tr(ad\,a\, ad\,b) = K(a,b)$

So g preserves $K(,)$ and hence $\langle , \rangle$.

$\square$

So, f is ind of the choice of H.