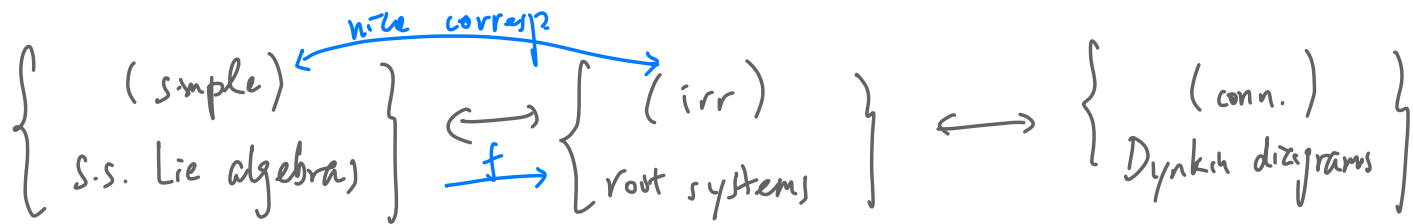
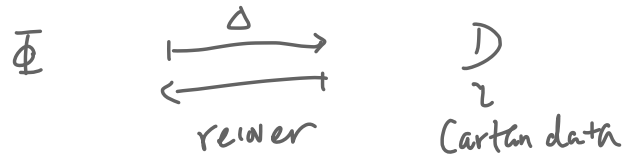


Last time: Almost completed studying the corresp.



$$f: \quad L + \underbrace{\text{a CSA } \mathfrak{H}}_{\text{choice}} \quad \xrightarrow{\quad} \quad \bar{\Phi} \quad \xleftarrow{\quad} \quad ?$$

DONE.



Today: A closer look at the left side.

Two questions:

1. Why is f independent of the choice of \mathfrak{H} ?
2. Why is f a bijection? In particular, how to recover L from $\bar{\Phi}$?

But first a missing proof.

Thm. Let L be a complex s.r. Lie algebra. Let H be a CSA of L , and let $\bar{\Phi}$ be the corr. root system. Then $L \triangleright \text{simple} \Leftrightarrow \bar{\Phi} \triangleright \text{irr.}$

Pf: (\Leftarrow). Suppose $\bar{\Phi} \triangleright \text{irr.}$ Consider the Cartan decomp.

$$L = H \oplus \bigoplus_{\alpha \in \bar{\Phi}} L_{\alpha} \quad (*)$$

Suppose that L is not simple. say L has a proper, nontrivial ideal $I \subseteq L$.

Recall that $L = \bar{I} \oplus I^{\perp}$. Since $H \triangleright \text{toral}$, each of I and I^{\perp} has an eigenbasis under the ^(w.r.t. Killing form) action $\text{ad } H$. Since $\dim L_{\alpha} = 1 \forall \alpha \in \bar{\Phi}$,

it follows that $I = H_1 \oplus \bigoplus_{\alpha \in \bar{\Phi}_1} L_{\alpha}$ and $I^{\perp} = H_2 \oplus \bigoplus_{\alpha \in \bar{\Phi}_2} L_{\alpha}$ for some

Subspaces H_1, H_2 of H and some subsets \bar{E}_1, \bar{E}_2 of \bar{E} s.t.

$\bar{E}_1 \cap \bar{E}_2 = \emptyset, \bar{E}_1 \cup \bar{E}_2 = \bar{E}$. We claim that $\bar{E}_1 \neq \emptyset, \bar{E}_2 \neq \emptyset$ and

$\bar{E}_1 \perp \bar{E}_2$ (i.e., $(\alpha, \beta) = 0 \forall \alpha \in \bar{E}_1, \beta \in \bar{E}_2$), This means $\bar{E} \ni$ reducible.

Contradiction.

To prove the claims, first note that if $\bar{E}_2 = \emptyset$ then $\bar{E}_1 = \bar{E}$. Since L
 \ni gen. by $\{L\alpha : \alpha \in \bar{E}\}$, it follows that $I = L$, contradicting properness of I .

So $\bar{E}_2 \neq \emptyset$. Similarly, $\bar{E}_1 \neq \emptyset$. Take $\alpha \in \bar{E}_1, \beta \in \bar{E}_2$. Let $\{e_\alpha, f_\alpha, h_\alpha\}$
and $\{e_\beta, f_\beta, h_\beta\}$ be the \mathbb{R}^3 -triples for α and β . Then $h_\beta = [e_\beta, f_\beta] \in I^\perp$.

$$s. \quad \alpha(h_\beta) e_\alpha = [h_\beta, e_\alpha] \in I \cap I^\perp = 0$$

s. $\alpha(h_\beta) \alpha = 0$. s. $\alpha(h_\beta) = (\alpha, \beta^\vee) = 0$.

This implies $(\alpha, \beta) = 0$ so $\mathbb{F}_1 \perp \mathbb{F}_2$, as desired.

(\Rightarrow). Suppose L is simple. Suppose, for contradiction, that \mathbb{F} is reducible. say
with $\mathbb{F} = \mathbb{F}_1 \cup \mathbb{F}_2$, $\mathbb{F}_1 \neq \emptyset$, $\mathbb{F}_1 \neq \mathbb{F}$. $\mathbb{F}_1 \cup \mathbb{F}_2 = \mathbb{F}$. $\mathbb{F}_1 \cap \mathbb{F}_2 = \emptyset$. $\mathbb{F}_1 \perp \mathbb{F}_2$.

For any $\alpha \in \mathbb{F}_1$, $\beta \in \mathbb{F}_2$,

$$(\alpha + \beta, \beta) = (\alpha, \beta) + (\beta, \beta) = (\beta, \beta) \neq 0 \text{ since } \beta \neq 0.$$

Similarly $(\alpha + \beta, \alpha) \neq 0$. s. $\alpha + \beta \notin \mathbb{F}_1 \cup \mathbb{F}_2 = \mathbb{F}$, i.e., $\alpha + \beta$ is not
a root, so $[\mathbb{L}_\alpha, \mathbb{L}_\beta] \in \mathbb{L}_{\alpha+\beta} = 0$, i.e. $[\mathbb{L}_\alpha, \mathbb{L}_\beta] = 0 \forall \alpha \in \mathbb{F}_1, \beta \in \mathbb{F}_2$.

Let L_1 be the subalgebra of L generated by $\{\mathbb{L}_\alpha : \alpha \in \mathbb{F}_1\}$; define L_2 similarly,

Note that:

- If $L_1 = L$, then $[L, L_2] = 0$, so $L_2 \subseteq Z(L)$, contradicting the simplicity of L .

See \downarrow Ex. Prop 14.2 for a more detailed proof.

- So $L_1 \subsetneq L$. But then $[L_1, L] = [L_1, L_1] \subseteq L_1$, so

L_1 is a proper, nonzero ideal in L , contradicting the simplicity of L .

Back to the questions on the map $f: (L, H) \mapsto \mathbb{F}_H$. □

Q 1. Why is f ind of the choice of H ?

Key fact: Fix L (s.s.). any two CSAs H and H' of L are conjugate under $\text{Inn}(L)$ in that $H' = \phi(H)$ for some $\phi \in \text{Inn}(L)$.

Conjugacy Thm. \leftarrow

(Recall that $\text{Inn}(L) = \langle \exp \text{ad } x : \text{ad } x \text{ nilp} \rangle \subseteq \text{Aut}(L)$)
 See Lecture 3

Remarks:

• The theorem appears as Thm 12.6, and is proved in Append. C.

m Ew. The proof partially "cheats" by using numerics in the exceptional types (using $\dim L = \dim \mathfrak{h} + |\Phi|$)

not too much machinery, but uses classification of root system

more conceptual,
 heavy machinery,
 doesn't need
 root system
 classification

• In turn, the theorem \Rightarrow treated in Ch 15 & Ch 16, which proves the stronger fact that any two Borel subalgebras of L are

conj under a certain subgroup $\mathcal{E}(L)$ of $\text{Inn}(L)$.
 (def: max. solvable subalg. see HW.)
 contains strongly ad-nilp elts)

Why does the conj. of H, H' imply $\mathfrak{F}_H \cong \mathfrak{F}_{H'}$?

Answer: Rough reason: say $H' = \phi(H)$. $\phi \in \text{Aut}(L)$.

Being an automorphism, ϕ preserves $[\cdot, \cdot]$, so it should "translate" Cartan decompositions. Moreover, ϕ respects the Killing form, so it preserves all geom. info about the root systems.

$$\alpha'(\phi(h)) = \alpha(h) \quad \forall h \in H.$$

(1) How to find an iso map $\mathfrak{F}_H \rightarrow \mathfrak{F}_{H'}$. every elt in \mathfrak{F}' is of the form

$H \rightarrow L = H \oplus \bigoplus_{\alpha \in \mathfrak{F}_H} L_\alpha$. Take $h \in H$, $\alpha \in \mathfrak{F}_H$, $x \in L_\alpha$. Then

$$[\phi(h), \phi(x)] = \phi([h, x]) = \phi(\alpha(h)x) = \alpha(h)\phi(x).$$

So $\phi(x)$ is in the α' -wt space of H' where $\alpha' \in \mathfrak{F}_{H'} \subseteq H'^*$ is defined by

Thus we have a map $g: \mathbb{F}H \rightarrow \mathbb{F}H'$, $\alpha \mapsto \alpha'$.

It's easy to see it's a bijection. (One proof: consider the obvious analog map from $\mathbb{F}H' \rightarrow \mathbb{F}H$, which will be a two-sided inverse).

②. Why g preserves \langle, \rangle .

Reason: $\forall a, b \in L, K(\phi(a), \phi(b)) = \text{Tr}(\text{ad } \phi(a) \text{ ad } \phi(b)) \stackrel{\text{HW}}{=} \text{Tr}(\text{ada ad b}) = K(a, b)$

So g preserves $K(,)$ and hence \langle, \rangle .

So, f is ind. of the choice of H . □