Last time: Classification of irr. root systems / conn. Dynkin diagrams.

(*) $A_n\ (n \geq 1)$, $B_n\ (n \geq 2)$, $C_n\ (n \geq 3)$, $D_n\ (n \geq 3)$, E_6, E_7, E_8, F_4, G_2

Two directions. I. Each lettered type can be indeed realized by a root system — construction.

II. A conn. Dynkin diagram has to be one of (*).

$\downarrow$

Introduced: admissible sets in E.

$A \longrightarrow$ admissible diagram.

vertices: A.

edges between a, b: $m_{ab} = 4(a,b)(a,b)$

Fact (easy): Φ a root system

$\Downarrow$ $\Leftarrow$

$A = \{ \underbrace{v / \sqrt{(v,v)}}_{v'} : v \in \Phi \}$ is admissible.

$m_{a'b'} = d_{ab} \leftarrow$ used in Dynkin.

Classification of conn. admissible diagrams:

(a) Establish "pattern avoidance" lemmas / props. (EW)

① Lemma 13.4. ("$|\mathcal{E}| < |\mathcal{V}|$") The number of adjacent pairs of vertices ($m_{ab} \neq 0$) is at most $|A| - 1$.

Pf: Say $A = \{v_1, \ldots, v_n\}$. Consider $v = \sum v_i$. Then $v \neq 0$ since A is lin. ind., so $(v, v) > 0$. But

$$(v, v) = \left(\sum v_i, \sum v_i\right) = \sum_i (v_i, v_i) + \sum_{i<j} 2(v_i, v_j) = n + \sum_{i<j} 2(v_i, v_j) \overset{>0}{}$$

So $|A| = n > \sum_{i<j} [-2(v_i, v_j)] = \sum_{i<j} \sqrt{m_{ij}} \geq$ # of adjacent pairs of vertices.
("$= |\mathcal{E}|$") □

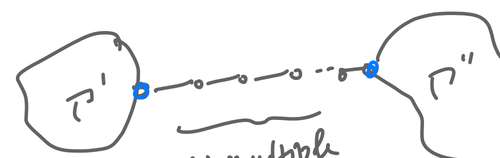
② **Coro 13.5.** (acyclicity) An admissible graph Γ cannot have a cycle.

Similar result.

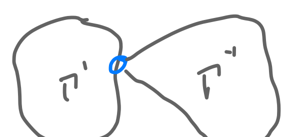
③ **Lemma 13.6.** (degree bound) Γ has no vertex of degree 4 or larger.

④ **Coro 13.7.** If Γ has a triple edge (= edge of wt. 3), then

Γ = 

⑤ **Lemma 13.8.** (shrinking lemma). If Γ = 

no multiple edges between any pair

is admissible, then 

is still admissible.

⑥ <u>Lemma 13.9.</u> (app. of shrinking lemma) An admissible graph Γ has

(i) no more than one double edge ;

(ii). no more than one branch vertex ;

(iii) not both a double edge and a branch vertex.

. . . . . .

more. eg. Prop 13.11. double bond in admissible $\Gamma \Longrightarrow \Gamma =$ 
or $\Gamma =$  .

1b) Putting together these restrictions narrows down the possibilities for $\overset{\text{conn.}}{\wedge}$ admissible diagrams. $\longrightarrow$ Connected Coxeter diagrams. A_n, $\underbrace{B_n / C_n}$, D_n, E_{6-8}, F_4, G_2.

(c) Put "orientations" on (the strong bonds) of the Coxeter diagram. B_n, C_n.

Done with the classification of irr. root systems / conn. Dynkin diagrams!

Back to Lie algebras:

Thm. (EW. Prop 12.4 + EW. Prop 14.2 / Hum 14.1.) Let L be a complex s.s. Lie algebra. Let H be a Cartan subalgebra and $\Phi \subseteq H^*$ be corresponding root system. (Φ. $E := \mathbb{R}\Phi$)

Then L is simple $\Longleftrightarrow$ Φ is irr.

Prop. (Hum. 14.1.) Let L be a complex s.s. Lie algebra. Let H be a Cartan subalgebra of L. and Φ its root system. If $L = L_1 \oplus \cdots \oplus L_t$ is the decomp. of L into simple ideals, then $H_i := H \cap L_i$ is a

Cartan subalgebra of L_i for all $1 \le i \le t$, and the root system Φ_i of L_i w.r.t. H_i may be regarded canonically as a subsystem of Φ, in such a way that $\Phi = \Phi_1 \cup \dots \cup \Phi_t$ is the decomp of Φ into its irreducible components.

How to make the identification $\Phi \longleftrightarrow \cup \Phi_i$:

$(\Leftarrow)$ $\alpha \in \Phi_i \subseteq H_i^* \longmapsto \alpha \in \Phi \subseteq H^*$ s.t. $\alpha(h) = \begin{cases} \alpha(h) & \text{if } h \in H_i \\ 0 & \text{if } h \in H_j, j \neq i \end{cases}$

$(\rightarrow)$ $\alpha \in \Phi \Rightarrow \exists!\, i$ s.t. $[H_i, L_\alpha] \neq 0$; have $\underline{\alpha(h) = 0 \text{ if } h \in H_j\ \forall j \neq i}$

naturally view $\underline{\alpha \in H_i^*}$.

By the theorem and proposition, we've now established correspondences

$$\left\{\begin{array}{c}\text{(simple)}\\ \text{s.s. Lie algebras}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{(irr.)}\\ \text{root systems}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{(conn.)}\\ \text{Dynkin diagrams}\end{array}\right\}$$

$$L \overset{H}{\underset{\substack{\text{Cartan}\\ \text{subalgebra}}}{\longmapsto}} \Phi \longmapsto D$$

Thus, we have classified f.d. complex semisimple Lie algebras

(modulo proving the fact that diff. choices of the Cartan subalgeba H yield the same root system $\longrightarrow$ technical, todo.)