Last time: · Root system of $sp(2\ell)$ (type C_ℓ)

· bij: $\left\{ \overset{(\text{irr})}{\text{root systems}} \right\} \big/ \text{iso} \longrightarrow \left\{ \overset{(\text{conn.})}{\text{Dynkin diagrams}} \right\}$

More about the decomp of root systems into irreducible ones. (so it suffices to study/classify irr. root systems / conn. Dynkin diagrams)

Prop: Let (E, Φ) be a root system. Then Φ can be written as a disjoint union $\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_k$ where for $E_1, \ldots, E_k$ s.t. $E_i := \operatorname{Span} \Phi_i$, the pair (E_i, Φ_i) is an irreducible root system and $E = E_1 \oplus \cdots \oplus E_k$ as a v.s.

Pf (sketch): Define an equivalence relation on Φ by declaring

$\alpha \sim \beta$ for $\alpha, \beta \in \Phi$ if $\exists$ a sequence

$$\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_l = \beta \quad \text{s.t.}$$

$(\alpha_i, \alpha_{i+1}) \neq 0$ $\forall 1 \leq i \leq l-1$. Take $\Phi_1, \ldots, \Phi_k$ to be the equiva. classes of $\sim$.

(so α_i, α_{i+1} have to lie in the same irr. comp of Φ / conn. component of the Dynkin diagram of Φ)

Claim: This gives the desired decomp.

Need to check: — Each class Φ_i is a root system in $E_i = \operatorname{Span} \Phi_i$.

(R1) ✓, (R2) ✓, (R3) ?, (R4) ✓

for checking (R3), it suffices to show that

$$(\alpha, \beta) \neq 0 \implies (\alpha, s_2(\beta)) \neq 0$$

Ex: Ew.11.2.

so $s_\alpha(\beta) \sim \alpha$ just as β is.

- Each R_i is irr. $\to$ clear as remarked.

- $E = E_1 \oplus \cdots \oplus E_k$.

(1) $E = E_1 + E_2 + \cdots + E_k = \operatorname{span} \Phi_1 + \cdots + \operatorname{span} \Phi_k$.

True since $\Phi = \bigcup \Phi_i$ spans E.

(2) If $v_1 + \cdots + v_k = 0$ then $v_j = 0 \ \forall j$.

To do this, pair with v_j for each j. $\square$

Connected Dynkin Diagrams

Note: Except for B_n & C_n, we can recover each type of diagram from its underlying Coxeter diagram. ("lengths are not that important.")

Thm (EW. 13.1) The Dynkin diagrams of irreducible root systems are exactly the following (mutually nonisomorphic) diagrams:

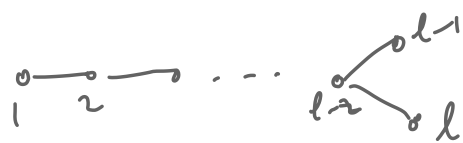
Inf. families:

A_ℓ $(\ell \geq 1)$

B_ℓ $(\ell \geq 2)$

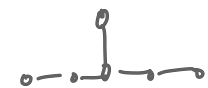
C_ℓ $(\ell \geq 3)$

D_ℓ $(\ell \geq 4)$



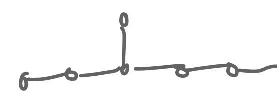
exceptional ones:

E_6



E_7

E_8



F_4

G_2

Pf (sketch): Two directions:

I. Construction: every diagram listed is the Dynkin diagram of an (irr) root system.

Method 1. realization via Cartan decomp of Lie algebras of the same type (e.g. we've seen sl_n for type A_{n-1} and $sp(2l)$ for C_l.)

Method 2. abstract constructions.

e.g. $A_{n-1} \longrightarrow \{ e_i - e_j : 1 \le i, j \le n, \ i \ne j \} \subseteq \mathbb{R}^n$.

types E_6, E_7, E_8, F_4, G_2, see EW.13.2.

II. A conn. Dynkin diagram has to be one of the listed diagrams.

Sketch of proof: ① Define admissible subsets of E and their corresponding diagram ("admissible diagrams").

vertices: $\leftrightarrow A$

edges: $m_{\alpha\beta} = 4(v_i, v_j)^2$

Def: A subset A of E is admissible if $A = \{v_1, \dots, v_n\}$

s.t. (a) A is lin ind

(b) $(v_i, v_i) = 1 \ \forall i \in [n]$. $(v_i, v_j) \le 0$ if $i \neq j$.

(c). If $i \neq j$. then $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$..

point: "lengths are not that important", so normalize every base vector first

Key fact: $\Phi \subseteq E$ is a root system, $\Delta \subseteq \Phi$ a base

$\Downarrow$

$A_\Phi := \left\{ \frac{\alpha}{\sqrt{(\alpha, \alpha)}} : \alpha \in \Delta \right\} \subseteq E$ is admissible, $m_{\alpha\beta} = d_{\alpha\beta}$ $\forall \alpha, \beta \in \Delta$.

Also note: Subsets of admi. sets are admissible.

(2). Establish "pattern avoidance" results on conn. admissible diagrams.

__Lemma 13.4__. ("$|\mathcal{E}| < |V|$") The number of vertices joined by at least one edge is at most $|A| - 1$.

Pf: next time.

So the diagram is tree!

__Coro 13.5.__ The diagram of A contains no cycles

__Pf:__ If there's a cycle C, its "support" violates "$|\mathcal{E}_c| < |V_c|$".

To be continued ...