<u>Last time:</u> - Root systems in $E = \mathbb{R}^2$

- Root system of $\mathfrak{sl}_n$ (type-A Lie algebra)

<u>Today.</u> I. Root system of $\mathfrak{sp}(2\ell, \mathbb{C}) =: \mathfrak{sp}_{2\ell}$ (type-C Lie algebra)

see 09.02.pdf, Hum. Ch 1, EW. 12.5.

can be defined from a bilinear form $\begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$.

Recall that

$$L := \mathfrak{sp}_{2\ell} = \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} : \begin{array}{l} m, p, q : \ell \times \ell \\ p = p^t, \ q = q^t \end{array} \right\}.$$

<u>Fact:</u> $H = \{ \text{diag. in } L \}$ is a Cartan subalgebra of L. (See EW Lemma 12.2)

So the Cartan decomp of L w.r.t. H gives a root system.

We'll compute the Cartan decomp.

$$H = L_0 = \underbrace{\langle}_{\mathbb{C}\text{-span}} e_{ii} - e_{(l+i)(l+i)} : 1 \leq i < l \rangle$$

<u>Define</u> $m_{ij} := e_{ij} - e_{l+j\, l+i}$ if $1 \leq i, j \leq l$

$p_{ij} = e_{i,l+j} + e_{j,l+i}$ for $1 \leq i \leq j \leq l$

$q_{ij} = e_{l+i,j} + e_{l+j,i}$ for $1 \leq i \leq j \leq l$

Then $L = H \oplus \bigoplus_{i,j} \langle m_{ij} \rangle \oplus \bigoplus_{i,j} \langle p_{ij} \rangle \oplus \bigoplus_{i,j} \langle q_{ij} \rangle$

as vector space spaces.

e.g. $l=2$, a typical matrix in L looks like

$$\left(\begin{array}{cc|cc} m_{11} & m_{12} & p_{11} & p_{12} \\ m_{21} & m_{22} & p_{12} & p_{22} \\ \hline q_{11} & q_{12} & -m_{11} & -m_{21} \\ q_{12} & q_{22} & -m_{12} & -m_{22} \end{array}\right)$$

and $\langle m_{ij}\rangle$, $\langle p_{ij}\rangle$, $\langle q_{ij}\rangle$ is a wt space whenever defined.

eg. $[h, m_{ij}] = (a_i - a_j)\, m_{ij} = (\varepsilon_i - \varepsilon_j)(h)\cdot m_{ij}$

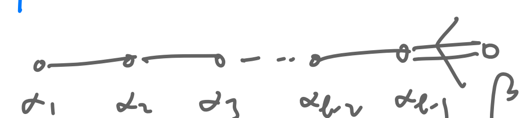
where $\varepsilon_k(\mathrm{diag}(a_1, \dots, a_\ell, -a_1, \dots, -a_\ell)) = a_k \quad \forall\ 1 \le k \le \ell$.

So we have the root system ①.

Ex: ① $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le \ell,\ i \ne j\} \cup \pm\{\varepsilon_i + \varepsilon_j \mid 1 \le i, j \le \ell\}$.

② $\Delta := \{\underbrace{\varepsilon_1 - \varepsilon_2}_{\alpha_1}, \dots, \underbrace{\varepsilon_{\ell-1} - \varepsilon_\ell}_{\alpha_{\ell-1}}, \underbrace{2\varepsilon_\ell}_{\beta}.\}$ is a base

③ The Dynkin diagram (for Δ) is



II. Dynkin diagrams.

Recall the def: $\Phi \subseteq E$ a root system. $\Delta \subseteq \Phi$ a base

- the graph D with vertices $\overset{bij}{\longleftrightarrow} \Delta$ (labels of vertices)

the graph without the arrow is called the Coxeter graph

- $\forall \alpha, \beta \in \Delta$. connect the vertices α, β with

$$d_{\alpha\beta} = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \text{ edges}$$

(in particular, α, β are not adjacent in $D \Leftrightarrow \alpha \perp \beta$ as roots, i.e. $(\alpha, \beta) = 0$)

- If α, β are adjacent and differ in length, then draw an arrow from the longer one to the shorter one.

Also recall: If Δ, Δ' are two bases of Φ, they give rise to $\cong$ graphs. $\left(\begin{matrix}\exists g \in W \\ \Delta' = g\Delta\end{matrix}\right)$

Prop 1. (Dynkin diagrams determine root systems) Let D_1, D_2 be the Dynkin diagrams of root systems $\Phi_1 \subseteq E_1$ and $\Phi_2 \subseteq E_2$. If D_1, D_2 are isomorphic as graphs, then $(E_1, \Phi_1) \cong (E_2, \Phi_2)$ as root systems.

Point: To classify root systems, it suffices to classify Dynkin diagrams.

$$\left(\text{root systems} \xleftrightarrow{\text{bij}} \text{Dynkin diagrams} \right) \text{ up to isos.}$$

Pf: By assumption, $\exists$ bij $\varphi : \Delta_1 \to \Delta_2$, say $\varphi(\alpha) = \alpha'$ $\forall \alpha \in \Delta_1$, such that $d_{\alpha\beta} = d_{\alpha'\beta'}$ $\forall \alpha, \beta \in \Delta_1$. Consequently, $\langle \alpha, \beta \rangle = \langle \alpha', \beta' \rangle$ $\forall \alpha, \beta \in \Delta_1$. Since Δ_1 is a basis of E_1, φ extends to a linear isomorphism $\varphi : E_1 \to E_2$ (abusing notation for φ here), s.t.

show that $(E_1, \Phi_1) \cong (E_2, \Phi_2)$ it suffices to show that $\varphi(\Phi_1) = \Phi_2$.

Note that $\forall v \in E_1$, say $v = \sum_{\alpha \in \Delta_1} c_\alpha \alpha$, we have

$$\langle v, \beta \rangle = \langle \varphi(v), \beta' \rangle \quad \forall \beta \in \Delta_1. \quad \text{(Ex.)}$$

and so $s_{\varphi(\beta)}(\phi(v)) = \phi(s_\beta(v))$, i.e., the diagram

$$\begin{tikzcd}
v \in E \arrow[r, "s_\beta"] \arrow[d, "\varphi"'] & E \arrow[d, "\varphi"] \\
E' \arrow[r, "s_{\varphi(\beta)}"'] & E'
\end{tikzcd}$$

commutes for all $\beta \in \Delta_1$.

Now take $\gamma \in \Phi_1$, then $\exists\ \beta_1, \beta_2 \cdots, \beta_k, \alpha \in \Delta$ s.t.

$$\gamma = s_{\beta_1} \circ \cdots \circ s_{\beta_k}(\alpha) \qquad \left(\text{Since } \Phi = W_0 . \Delta . \right)$$

So using the comm diagram repeatedly for $\beta_1, \cdots \beta_k$, we have

$$\varphi(\gamma) = \varphi\left(s_{\beta_1} \cdots s_{\beta_k}(\alpha)\right) = \underbrace{s_{\beta_1'} \circ \cdots \circ s_{\beta_k'}}_{\in W_2} \cdot \underset{\in \Delta_2}{\alpha'} \in \Phi_2$$

Thus, we have $\varphi(\Phi_1) \subseteq \Phi_2$. We may apply the same argument to $\varphi^{-1}: E_2 \to E_1$, $\alpha' \mapsto \alpha$ $\forall \alpha' \in \Delta_2$. to obtain $\varphi^{-1}(\Phi_2) \subseteq \Phi_1$, so $\Phi_2 \subseteq \varphi(\Phi_1)$. It follows that $\Phi_2 = \varphi(\Phi_1)$. $\square$

Next time: classify root systems via Dynkin diagrams

There's a refinement of the bij $\left\{ \begin{array}{c} \text{root systems} \\ \text{up to iso} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Dynkin diag.} \\ \text{up to iso} \end{array} \right\}$.

namely, a root system is "irreducible" iff its Dynkin diagram is connected.

Def: A root system Φ is <u>irr</u> if it cannot be partitioned into

$\Leftrightarrow$ a disjoint union of nonempty sets $\Phi = \Phi_1 \sqcup \Phi_2$ s.t. (1) Φ_1, Φ_2 are root systems themselves and (2) $(\alpha, \beta) = 0 \quad \forall \alpha \in \Phi_1, \beta \in \Phi_2$.

Prop. Every root system can be decomposed into irreducibles.

(EW.11.7) (so it suffices to study irr. root systems, which clearly correspond to connected Dynkin diagrams)

Upshot: Studying conn. Dynkins is enough.