Last time: Action of Weyl gps on Weyl chambers and bases.

- The two actions are compatible with the bijection between chambers and bases, and both actions are faithful and transitive.

Today. Examples: root systems of rank 2 and type ABCD.

I. Classification of root systems in $E = \mathbb{R}^2$.

Let Φ be a root system in E and let $\Delta = \{\alpha, \beta\}$ be base of Φ. say with $\|\beta\| \geq \|\alpha\|$.

(1) Recall that by 'the table', we must have $(\alpha, \beta) \leq 0$, for otherwise $(\alpha, \beta) > 0$ and $\underset{+}{\alpha} - \underset{-}{\beta} \in \Phi$, so Δ would not be a base.

So the relevant portion of the table is ...

$(\alpha,\beta^\vee)(\beta,\alpha^\vee)=4\cos^2\theta$	θ	$(\alpha,\beta^\vee)$	$(\beta,\alpha^\vee)$	$\frac{\|\beta\|}{\|\alpha\|}$
0	$\frac{\pi}{2}$	0	0	undetermined
1	$\frac{2\pi}{3}$	-1	-1	1
2	$\frac{3\pi}{4}$	-1	-2	$\sqrt{2}$
3	$\frac{5\pi}{6}$	-1	-3	$\sqrt{3}$

(2). If c is a scaling map and r is a rotation, then $c\Phi'$ and $r\Phi'$ are root systems whenever Φ' is a root system (in E).

So we may assume that $\alpha = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Similarly, we may assume $\beta \in [0, \pi]$ by reflection considerations.

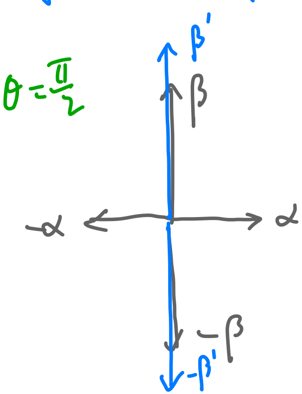
(3). Recall that $\Phi = W_0 \Delta = \langle s_\alpha : \alpha \in \Delta \rangle \cdot \Delta$.

Prop: $\forall \beta \in \Phi$, $\exists g \in W_0$, $\alpha \in \Delta$ s.t. $\beta = g(\alpha)$.

So, Φ can be recovered from the simple roots and simple reflections alone.

The classification. (4 types depending on θ)

(a). $\theta = \frac{\pi}{2}$

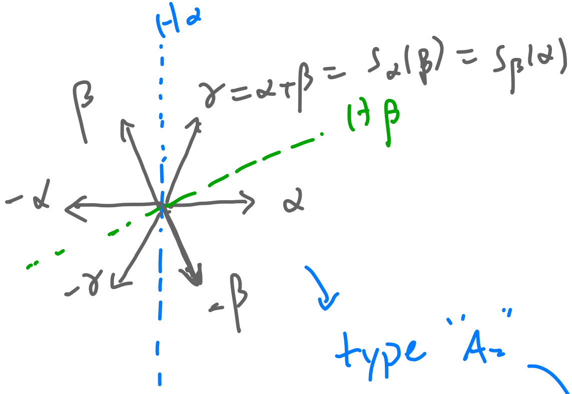


The ratio $\frac{\|\beta\|}{\|\alpha\|}$ can be an arbitrary pos. scalar.

$\to$ type "$A_1 \times A_1$"

$\{\alpha, -\alpha\}$ $\quad$ $\{\beta, -\beta\}$

(b) $\theta = \frac{2\pi}{3}$ $\qquad \frac{\|\beta\|}{\|\alpha\|} = 1.$



type "A_2"

(E, Φ) is iso to root system arising from the Cartan decomp of $\mathfrak{sl}_3$, the Lie algebra of type A_2

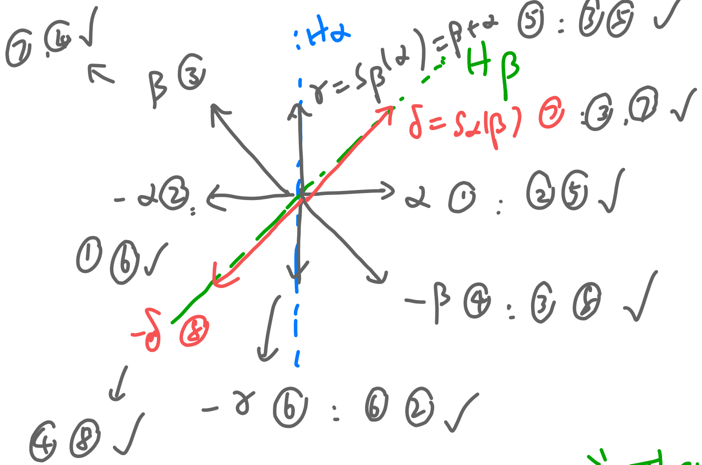
$$\{ \varepsilon_i - \varepsilon_j \mid 1 \le i, j \le 3,\ i \ne j \} \subseteq H^*$$

Two ways to see we've exhausted all roots in the system.

Method 1: another root δ would form an angle smaller than $\frac{\pi}{6}$ with one of $\pm\alpha, \pm\beta, \pm\gamma$ (Pigeonhole principle!), which is not allowed by 'the' (complete) table.

Method 2. check invariance under s_α, s_β. (more in the next eg.)

(c). $\theta = \frac{3\pi}{4}$. $\frac{\|\beta\|}{\|\alpha\|} = \sqrt{2}$

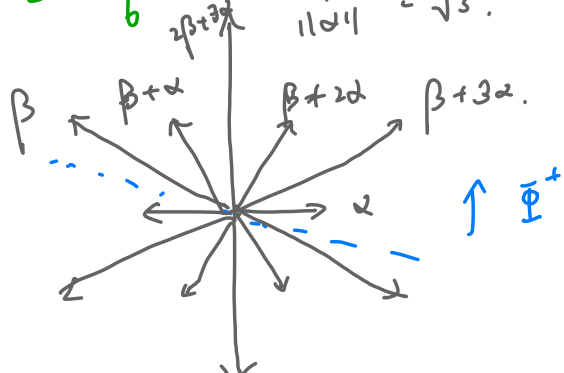


We'll check for completeness as we draw the roots $\longrightarrow$ we'll know when to stop.

$\searrow$ type "B_2". $\leftarrow$ Is to the root system arising from the Lie algebra of type B_2

"They are exactly the crystallographic dihedral gp."

(d). $\theta = \frac{5\pi}{6}$ $\frac{\|\beta\|}{\|\alpha\|} = \sqrt{3}$.



Note: In cases (b), (c), (d), the Weyl gp is generated by two involutions s_α, s_β; and the order of $s_\alpha s_\beta$ (a rotation, angle dependent on θ : $2(\pi - \theta)$) is 3, 4 and 6, resp.

By general gp theory, W is $I_2(3)$, $I_2(4)$, $I_2(6)$. dihedral

Realizations of root systems of type ABCD via Lie algebras.

Ex: Let $L = sl_{n+1}$, the Lie algebra of type A_n.

Recall that L gives rise to a root system (say, with $H = \{\text{diag. in } L\}$)

$$\Phi_n = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n+1, \; i \neq j \}. \subseteq E_n := \mathrm{Span}(\Phi).$$

Prove that (E_2, Φ_2) is iso to the root system $(E = \mathbb{R}^2, \Phi)$ from (b).

Ex: Recall the symplectic Lie algebra $sp(2l, \mathbb{C})$, the Lie algebra of type C_l. Prove that the Cartan decomp of B_l (say, $H = \{\text{diag. in } L\}$) gives a root system iso to the dual of the one in (c) when $l = 2$.

next time: general l. C_l.