Last time: E, Φ, Δ (fixed base)

$\downarrow$

Weyl gp W.

Def: $W_0 = \langle s_\alpha : \alpha \in \Delta \rangle \subseteq W$.

- **Thm:** W is finite.

- **Thm:** $W = W_0$. $\leftarrow$ Suffices to prove: $\forall \beta \in \Phi, \exists g \in W_0, \alpha \in \Delta$ s.t. $\beta = g(\alpha)$.

Pf: Suffices to consider $\beta \in \Phi^+$. (if $\beta \in \Phi^-$, then $-\beta \in \Phi^+$, so get $g' \in W_0$, $\alpha \in \Delta$ s.t. $g'(\alpha) = -\beta$ so $g' \underbrace{s_\alpha(\alpha)}_{-\alpha}$ so $(g := g' s_\alpha, \alpha)$ works.)

For $\beta \in \Phi^+$, use induction on $ht(\beta)$

(Recall: $ht\left(\sum_{\gamma \in \Delta} k_\gamma \gamma\right) = \sum k_\gamma$.)

If $ht(\beta) = 1$, $\beta \in \Delta$, so $g = e$ works.

If $ht(\beta) > 1$. Say $\beta = \sum_{\gamma\in\Delta} k_\gamma \gamma$ with

① $k_\gamma \geq 0 \ \forall \gamma \in \Delta$, and $k_\gamma > 0$ for some $\gamma \in \Delta$.

then since ② $(\beta, \beta) = \sum_{\gamma\in\Delta} k_\gamma (\gamma, \beta) > 0$ ($(\beta,\beta) \neq 0$ since $\beta \neq 0$),

we must have $(\gamma_0, \beta) = (\beta, \gamma_0) > 0$ for some $\gamma_0 \in \Delta$.

Consider $s_{\gamma_0}(\beta) = \beta - \underbrace{\langle \beta, \gamma_0 \rangle}_{>0} \gamma_0$.

Recall that s_{γ_0} permutes $\Phi^+ \setminus \{\gamma_0\}$, so $s_{\gamma_0}(\beta) \in \Phi^+$.

On the other hand, we must have $ht(s_{\gamma_0}(\beta)) < ht(\beta)$ since $(\beta, \gamma_0) > 0$.

By induction, $\exists\, h \in W_0$ s.t. $s_\alpha(\beta) = h(\alpha)$ for some $\alpha \in \Delta$.

It follows that $\beta = s_\alpha(s_\alpha(\beta)) = s_\alpha h(\alpha) = g(\alpha)$ for $g := s_\alpha h \in W_0$. Done. $\square$

Action of the Weyl gp on Weyl chambers and Bases (E, Φ).

Recall the bijection $\{\text{Weyl chambers of } E\} \longleftrightarrow \{\text{Bases of } \Phi\}$

$\Vert$

conn. comp of $E \setminus \underbrace{\bigcup_{\alpha \in \Phi} H_\alpha}$

regular elts

Notation: If γ is regular, we denote the chamber it's in by $C(\gamma)$.

$$C \xrightarrow[\text{any}]{\text{take}} \gamma \in C \longmapsto \Delta(\gamma) : \begin{Bmatrix} \text{indecomp. on} \\ \text{the positive} \\ \text{side of } \gamma \end{Bmatrix}$$

$$C(\gamma) := \left\{ \text{regular } \gamma : \begin{matrix} (\gamma, \alpha) > 0 \\ \forall \alpha \in \Delta \end{matrix} \right\} \longleftarrow\!\!| \quad \Delta$$

Upshot: "W acts on both sides and the actions are compatible.

Moreover, both actions are faithful and transitive."

Observe that: Let $g \in W$.

(1) If Δ is a base of Φ, then so is

$$\Delta' := g(\Delta) := \{ g(\alpha) : \alpha \in \Delta \}.$$

$\to$ So W acts on the set of bases of Φ.

Pf: Need to check the base axioms for Δ'.

(B1) Δ' should be a basis. $\to$ True: Δ is a basis, and $g \in GL(E)$ is an automorphism of E.

(B2) Need $\Phi \subseteq \sum_{\alpha \in \Delta'} \mathbb{Z}_{\geq 0} \alpha \cup \sum_{\alpha \in \Delta'} \mathbb{Z}_{\leq 0} \alpha$.

$\downarrow$

True: g is linear, so it preserves linear comb.

$\forall \beta \in \Phi$, $\exists k_\gamma, \gamma \in \Delta$ s.t. $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$, $k_\gamma \geq 0 \ \forall \gamma$ or $k_\gamma \leq 0 \ \forall \gamma$.

$g(\beta) = \sum_{\gamma \in \Delta} k_\gamma \, g(\gamma) \subseteq \sum_{\alpha' \in \Delta} \mathbb{Z}_{\geq 0} \alpha' \cup \sum_{\alpha' \in \Delta'} \mathbb{Z}_{\leq 0} \alpha'$

all elts in Φ can be written this way.

(2). If γ is regular, then so is $g(\gamma)$.

Pf: $\forall \alpha \in \Phi$. then $\alpha = g(\beta)$ for some $\beta \in \Phi$ $\left(\text{take } \beta = g^{-1}(\alpha)\right)$.

so $\big(g(\gamma), \alpha\big) = \big(g(\gamma), g(\beta)\big) = (\gamma, \beta) \neq 0$ since γ is regular.

So $g(\gamma)$ is regular. $\square$

· If C is a Weyl chamber, then so is $g(C)$.

Pf: Suffices to consider $g = s_\alpha$, $\alpha \in \Phi$.

(a) Take $\gamma \in C$ so that $C = C(\gamma)$. Then $g(\gamma)$ is regular and $g(\gamma) \in g(C)$.

(b) g is a reflection, so it's continuous, so it maps conn. sets to connected sets, so $g(C)$ lies in a unique chamber, necessarily $C(g(\gamma))$. i.e. $g(C) \subseteq C(g(\gamma))$.

1c) $g^2 = s_\alpha^2 = e$, so

$$C(g(\gamma)) = \underline{gg\,C(g(\gamma))} \overset{1b)}{\subseteq} g\,C(gg(\gamma)) = g\,C(\gamma) = gC.$$

So $gC = C(g(\gamma))$. $\square$

(3). The actions are compatible: $g(C(\Delta)) = C(g(\Delta))$ for every base Δ of Φ.

Pf: Take $\gamma \in C(\Delta)$. Then $g(C(\Delta)) \ni g(\gamma)$. So it suffices to show that $g(\gamma) \in C(g(\Delta))$. This is true since

$$(g(\gamma), g(\alpha)) = \underbrace{(\gamma, \alpha) > 0 \;\forall \alpha \in \Delta}_{\text{since } \gamma \in C(\Delta)}.$$

$\square$

Harder results:

Thm 1. Let $S_\Delta = \{ s_\alpha : \alpha \in \Delta \}$ for each base Δ of Φ.

Then (W, S_Δ) is a Coxeter system for every base Δ of Φ.

can be used to prove

Consequence: · Elts of W have length : $\forall w \in W$ $\ell(w) \overset{\text{def.}}{=} \left(\begin{array}{l} \min \ell \text{ s.t. } w = s_1 \cdots s_\ell \\ \text{for some } s_1, \dots, s_\ell \in S_\Delta \end{array} \right)$.

· $\ell(w) \overset{\text{prop.}}{=} \left| \{ \alpha \in \Phi_\Delta^+ \mid w(\alpha) \in \Phi_\Delta^- \} \right|$ $\forall w \in W$.

Thm 2. W acts simply transitively on the set of bases of Φ, i.e.,

① Given any two bases Δ, Δ' of Φ, $\exists g \in W$ s.t. $\Delta' = g(\Delta)$. (transitivity)

② For any base Δ of Φ, if $g(\Delta) = \Delta$ for some $g \in W$, then $g = e$. ("simply")

Coro 1: W acts simply transitively on the Weyl chambers.

Pf: Thm 2 + Ob. (3).

Ex: Formulate and prove the corollary more carefully

Coro 2: We can define a well-defined graph for Φ via any base Δ of Φ which has vertex set Δ and with

"The Dynkin diagram of Φ."

$$d_{\alpha\beta} := \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$$

edges/lines between α and β $\forall \alpha, \beta \in \Delta$. (We'll do this more carefully soon)

Well-defined: difference bases give the same graph. (up to relabeling vertices) $\checkmark$

Pf: $\Delta, \Delta' \to \exists g \in W$ $\Delta' = g(\Delta)$. $d_{g(\alpha) g(\beta)} = d_{\alpha\beta}$ since g preserves $\langle , \rangle$. $\square$