Last time: Bijection $\{\text{Weyl chambers of } E\} \longleftrightarrow \{\text{bases of } \Phi\}$.

$\Phi \subseteq E$

Today: Weyl gp : subgp W of $GL(E)$ generated by the reflections $s_\alpha : \beta \mapsto \beta - (\beta, \alpha^\vee)\alpha = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$
$(\alpha \in \Phi)$

We'll see (*) $W = \langle s_\alpha : \alpha \in \Delta \rangle$ for any base Δ of Φ.

Next time: (perhaps) Action of Weyl gp on chambers and bases.

Preparation for (*): Recall that $W \overset{\text{def}}{=} \langle s_\alpha : \alpha \in \Phi \rangle \subseteq GL(E)$

Fix a base Δ of Φ.

Recall $\langle v, w \rangle = (v, w^\vee)$

$= \frac{2(v,w)}{(w,w)}$

$\forall v, w \in E, w \neq 0.$

(1). **Prop:** $s_\alpha^2 = 1_E \quad \forall \alpha \in \Phi.$

Pf 1. "of course, reflections are involutions."

Pf 2. $\forall v \in E,$

$$s_\alpha s_\alpha(v) = s_\alpha(v - \langle v, \alpha \rangle \alpha)$$

$$= v - \langle v, \alpha \rangle \alpha - \langle v, \alpha \rangle (-\alpha)$$

$$= v. \qquad \square$$

Coro: s_α permutes $\Phi \quad \forall \alpha \in \Phi.$

Pf: 1). s_α is inv since $s_\alpha^2 = 1_E$. 2). $s_\alpha(\Phi) \subseteq \Phi$ by axiom. $\square$

(2). By our proof/construction of bases of Φ, we have $\Phi^+ \subseteq \sum_{\alpha \in \Phi} \mathbb{Z}_{\geq 0} \cdot \alpha$ (and hence $\Phi^- \subseteq \sum_{\alpha \in \Phi} \mathbb{Z}_{\leq 0} \alpha$). In particular, for any $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma \in \Phi^+,$

we may define the <u>height</u> of β to be the integer

$$ht(\beta) := \sum_{\gamma \in \Delta} k_\gamma .$$

Heights are often useful in inductive proofs.

(3) **Prop**: s_α "preserves" $(\ ,\ )$ and hence $\langle\ ,\ \rangle$ and angles, in the sense

that $\big(s_\alpha(\beta), s_\alpha(\gamma)\big) = (\beta,\gamma)$ and $\langle s_\alpha(\beta), s_\alpha(\gamma)\rangle = \langle \beta, \gamma\rangle \ \forall \beta,\gamma \in \Phi$.

pf:
$$(s_\alpha(\beta), s_\alpha(\gamma)) = \left(\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha,\ \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha\right) = (\beta,\gamma) - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}(\alpha,\gamma)$$
$$- \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}(\beta,\alpha) + \frac{4(\beta,\alpha)(\gamma,\alpha)}{(\alpha,\alpha)\cancel{(\alpha,\alpha)}}\cancel{(\alpha,\alpha)} = (\beta,\gamma).$$

$$\langle s_\alpha(\beta), s_\alpha(\gamma)\rangle = \frac{2\big(s_\alpha(\beta), s_\alpha(\gamma)\big)}{(s_\alpha(\gamma), s_\alpha(\gamma))} = \frac{2(\beta,\gamma)}{(\gamma,\gamma)} = \langle \beta, \gamma \rangle. \qquad \square$$

(4). **Prop:** $\forall \alpha \in \Phi$, $g \in W$. we have $g s_\alpha g^{-1} = s_{g(\alpha)}$.

Pf: $\forall \beta \in \Phi$. $\quad g s_\alpha g^{-1}(\underbrace{g(\beta)}_{\gamma}) = g s_\alpha(\beta)$

Define $\gamma := g(\beta)$

$$= \begin{cases} \underbrace{g(\beta)}_{\gamma} & \text{if } \beta \in H_\alpha \quad (\text{equiv}^1 \text{ if } \gamma \in H_{g(\alpha)}) \\ g(-\beta) = \underbrace{-g(\beta)}_{-\gamma} & \text{if } \beta = \alpha. \quad (\text{equiv}^2 \text{ if } \gamma = g(\alpha)) \end{cases}$$

better pf

(Equivalence 1 holds since $(g(\alpha), g(\beta)) = (\alpha, \beta)$, so $\beta \in H_\alpha \iff g(\beta) = \gamma \in H_{g(\alpha)}$;

Equivalence 2 holds since $\beta = \alpha \iff \gamma = g(\beta) = g(\alpha)$ since g is inv.)

Thus, $g s_\alpha g^{-1}$ fixes $H_{g(\alpha)}$ and sends $g(\alpha)$ to $-g(\alpha)$. So $g s_\alpha g^{-1} = s_{g(\alpha)}$. □

Pf': $\forall \gamma \in E$, $g s_\alpha g^{-1}(\gamma) = g\left(g^{-1}(\gamma) - \underbrace{\langle g^{-1}(\gamma), \alpha \rangle}_{\substack{\| \\ \langle \gamma, g(\alpha) \rangle \text{ by (3)}}} \alpha\right) = \gamma - \underline{\langle \gamma, g(\alpha) \rangle}\, g(\alpha) = s_{g(\alpha)}(\gamma)$. □

Theorems on Weyl gps.

Thm 1. W is finite.

Pf: Since W acts on Φ by permutation, we have a gp hom

$$\begin{array}{ccc} f: W & \longrightarrow & \mathrm{Symm}(\Phi) \\ \cap & & \downarrow \\ GL(E) & & \text{symm. gp on } \Phi \text{, finite since } |\Phi| < \infty. \end{array}$$

So it suffices to show that W is inj. Say $f(w)$ fixes Φ $\underbrace{\text{for some}}_{\text{pointwise}}$ $w \in W$.

Since Φ spans E, $f(w) = id_E$ as a map on E. therefore f is inj.

$\square$

Thm 2. $W = \langle \underbrace{s_\alpha : \alpha \in \Delta}_{\text{simple reflections}} \rangle .$

First we prove a lemma and a prop.

Lemma. $\alpha \in \Delta \Rightarrow s_\alpha$ permutes $\Delta^+ \setminus \{\alpha\}$.

Pf: Take $\beta \in \Phi^+ \setminus \{\alpha\}$. Then $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ where $k_\gamma > 0$ for <u>some</u> $\gamma \in \Delta \setminus \{\alpha\}$. Now consider

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \sum k_\gamma \gamma - \langle \beta, \alpha \rangle \alpha .$$

In the expansion of the difference in terms of Δ, k_γ remains the same, with $k_\gamma > 0$. So $s_\alpha(\beta) \in \Phi^+$. Also, $s_\alpha(\beta) \neq \alpha$ since otherwise $\beta = -\alpha$. $\square$.

(Aside: **Coro:** Let $\rho = \frac{1}{2}\sum_{\gamma\in\Phi^+}\gamma \in E$. Then $s_\alpha(\rho) = \rho - \alpha$ $\forall \alpha\in\Delta$.

Pf: $\rho = \frac{1}{2}\alpha + \frac{1}{2}\sum_{\beta\in\Phi^+\setminus\{\alpha\}}\beta$. By the lemma,

$$s_\alpha(\rho) = -\frac{1}{2}\alpha + \frac{1}{2}\sum_{\beta\in\Phi^+\setminus\{\alpha\}}\beta = \rho - \alpha.$$)

Prop: Let $\beta\in\Phi$. Then $\exists\, g\in \underbrace{\langle s_\gamma : \gamma\in\Delta\rangle}_{W_0} \leq W$ and $\alpha\in\Delta$ s.t. $\beta = g(\alpha)$.

Note that Thm 2 follows immediately from the prop:

Pf: Let $W_0 = \langle s_\gamma : \gamma\in\Delta\rangle$. We want to show $s_\beta\in W_0$ for all $\beta\in\Phi$. By the prop and observation (4),

$$s_\beta = s_{g(\alpha)} = g s_\alpha g^{-1} \text{ for some } g\in W_0 \text{ and } \alpha\in\Delta. \text{ So } s_\beta\in W_0. \quad \square$$