Last time: - Pairs of roots

- Base: A base of a root system (E, Φ) is a basis[1] Δ of E s.t.[2] α is either a nonnegative lin. comb of the elts of Δ or a nonpos. lin comb of the elt of Δ.

Thm: Every root system (E, Φ) has a base.

Pf: $\dim E = 1$: obvious. $\dim E \geq 2$: take any $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$,

define $\Phi_\gamma^+ = \{\alpha \in \Phi ; (\alpha, \gamma) > 0\}$, $\Phi_\gamma^- = \{\alpha \in \Phi : (\alpha, \gamma) < 0\}$

and let $\Delta(\gamma) = \{$indecomposables in $\Phi_\gamma^+\}$.

Claim: $\Delta(\gamma)$ is a base. (not a sum $\gamma = \alpha' + \alpha''$ where $\alpha', \alpha'' \in \Phi_\gamma^+$.

Pf of the claim: Say $\Delta(\gamma) = \{\alpha_1, \ldots, \alpha_\ell\}$.

① Every elt of Φ_γ^+ is a nonnegative sum of $\alpha_1, \ldots, \alpha_\ell$.

Pf: Suppose not. Pick $\alpha \in \Phi_\gamma^+$ that is not a non neg. l.n. comb of $\alpha_1, \ldots, \alpha_\ell$ with (α, γ) minimal. In particular, $\alpha \notin \Delta(\gamma)$, so α is decomposable so $\alpha = \alpha' + \alpha''$ where $\alpha', \alpha'' \in \Phi_\gamma^+$. Thus, $(\alpha, \gamma) = \underbrace{(\alpha', \gamma)}_{>0} + \underbrace{(\alpha'', \gamma)}_{>0}$

By the minimality assumption, since $(\alpha, \gamma) > (\alpha', \gamma)$ and $(\alpha, \gamma) > (\alpha'', \gamma)$, we must have that α', α'' are nonneg. lin comb of Δ. But then so is α. Contradiction. $\square$

Corollary: Δ spans Φ and hence E, and Δ satisfies (2).

② We'll show that $(\alpha_i, \alpha_j) \leq 0$ for all $1 \leq i \neq j \leq \ell$.

Pf: If not, $(\alpha_i, \alpha_j) > 0$, so $\alpha_i - \alpha_j \in \Phi$, so

$\alpha_i - \alpha_j \in \Phi^+$ or $\alpha_j - \alpha_i \in \Phi^+$.

Say $\alpha_i - \alpha_j \in \Phi^+$, then $\alpha_i = \underbrace{(\alpha_i - \alpha_j)}_{\in \Phi^+_\gamma} + \underbrace{\alpha_j}_{\in \Phi^+_\gamma}$ This

contradicts the fact that α_i is indecomp.

Similarly, if $\alpha_j - \alpha_i \in \Phi^+$, then $\alpha_j = \underbrace{(\alpha_j - \alpha_i)}_{\in \Phi^+_\gamma} + \underbrace{\alpha_i}_{\in \Phi^+}$, which

contradicts the fact that α_j is indecomp.

③ Δ is lin. ind.

Pf: Say $\sum_{i=1}^{l} \underbrace{r_i}_{\in \mathbb{R}} \alpha_i = 0$. Separate $\{r_1, \ldots, r_l\}$ by

sign: $\sum_{\substack{r_i \geq 0 \\ i \in I}} r_i \alpha_i \overset{\varepsilon}{\overset{\|\mathrm{def}}{=}} \sum_{\substack{r_j' \geq 0 \\ j \in J}} r_j' \alpha_j$ with $I \sqcup J = \{1, \ldots, l\}$.

We have $(\varepsilon, \varepsilon) = \left(\sum r_i \alpha_i, \sum r_j' \alpha_j\right) = \sum_{i,j} \underbrace{r_i r_j'}_{\geq 0} \underbrace{(\alpha_i, \alpha_j)}_{\leq 0} \leq 0$

It follows that $\varepsilon = 0$. So

$$0 = (\gamma, \underset{\substack{\| \\ \sum r_i \alpha_i}}{\varepsilon}) = \sum \underbrace{\underbrace{r_i}_{\geq 0} \underbrace{(\gamma, \alpha_i)}_{> 0}}_{\geq 0} \Longrightarrow r_i = 0 \ \forall i \in I.$$

Similarly. $r_j' = 0$ and hence $r_j = 0$ $\forall j \in J$. $\square$

Prop: Any base Δ of Φ is of the form $\Delta(\gamma)$ for some $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

(So the map $\Delta : E \setminus \bigcup H_\alpha \to$ Bases of Φ, $\gamma \mapsto \Delta(\gamma)$ is surj.)

Pf: Given Δ, we may select $\gamma \in E$ s.t. $(\gamma, \alpha) > 0 \ \forall \alpha \in \Delta$:

- Use Gram-Schmidt to get an orthonormal basis $\beta = \{e_1, \dots, e_\ell\}$.

Then $\forall \alpha \in \Delta$, $(\gamma, \alpha) = [\gamma]_\beta \cdot [\alpha]_\beta$.

Take any $c_1, \dots, c_\ell > 0$, To find γ s.t. $(\gamma, \alpha_i) = c_i$ is then to solve

$$\begin{bmatrix} [\alpha_1]_\beta^t \\ \vdots \\ [\alpha_\ell]_\beta^t \end{bmatrix} \begin{bmatrix} \gamma \end{bmatrix}_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_\ell \end{bmatrix} \text{ for } [\gamma]_\beta.$$

invertible since Δ is a basis of E, so there is a solution.

Claim: $\Delta = \underbrace{\Delta(\gamma)}_{\text{indecomp. in } \Phi_\gamma^+}$.

Pf: Note that γ is regular, i.e. $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

Also, since $(\gamma, \alpha) > 0$ $\forall \alpha \in \Delta$, we have $\underbrace{\Phi_\Delta^+}_{\text{roots in } \Delta \text{ that are nonneg. lin comb of } \Delta} \subseteq \Phi_\gamma^+$ and $\Phi_\Delta^- \subseteq \Phi_\gamma^-$.

But $\Phi_\Delta^+ \cup \Phi_\Delta^- = \Phi_\gamma^+ \cup \Phi_\gamma^- = \Phi$,

So $\Phi_\Delta^+ = \Phi_\gamma^+$, $\Phi_\Delta^- = \Phi_\gamma^-$.

Next, note that $\Delta \subset \Delta(\gamma)$, i.e. x is indecomp in Φ_γ^+ for all $x \in \Delta$.

Pf: Say $\Delta(\gamma) = \{\alpha_1, \dots, \alpha_\ell\}$, $\Delta = \{x_1, \dots, x_\ell\}$. Take $x_j \in \Delta \subseteq \Phi_\Delta^+ = \Phi_\gamma^+$, so

$x_j \overset{*}{=} \sum_{i=1}^{\ell} c_i \alpha_i$. $c_i \geq 0$ $\forall i$. Each α_i is a nonneg lin comb of the x_i's.

Expand the right side of $(*)$ into a lin comb of Δ.

(positivity)

Since there's no cancellation, no α_i can have a x_k-comp if $k \neq j$.

So α_i is a mult. of x_j. But $\langle x_j \rangle \cap \Phi = \{\pm x_j\}$.

and $\Delta(\gamma)$ is a basis. So the sum on the right side

can only contain one summand, and it has to be x_j.

This proves $\Delta \subseteq \Delta(\gamma)$.

Since $|\Delta| = |\Delta(\gamma)| = \dim E = \ell$, it follows that $\Delta = \Delta(\gamma)$. $\square$

Def: Call each connected component of $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ a **Weyl chamber**.

Thus, each $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ lies in a unique Weyl chamber; we denote it by $C(\gamma)$.

Note: $C(\gamma) = C(\gamma') \iff (\gamma, \alpha)$ and (γ', α) have the same sign $\forall \alpha \in \Phi$.

for γ, γ' regular $\iff \Phi^+_\gamma = \Phi^+_{\gamma'}$

$\iff \Delta(\gamma) = \Delta(\gamma')$

take indecomp. / take nonneg span $(\cap \Phi)$

So, we have a well-defined bij. $\Delta : \left\{ \begin{array}{c} \text{Weyl chambers in} \\ E \setminus \bigcup_{\alpha \in \Phi} H_\alpha \end{array} \right\} \longrightarrow \{\text{bases of } \Phi\}$

$C \longmapsto \Delta(\gamma)$ for any $\gamma \in C$.