Last time: · We'll classify s.s. Lie algebras via classification of root systems.

· first observations on root systems.

Today.

1. **Pairs of roots** (E, Φ) $\alpha, \beta \in \Phi$ $\alpha \neq \pm\beta$.

Note: By the root system axioms, $(\mathbb{R}\alpha \oplus \mathbb{R}\beta) \cap \Phi$ form a root system in $\mathbb{R}\alpha \oplus \mathbb{R}\beta$.

Recall: With $\langle \gamma, \delta \rangle := \dfrac{2(\gamma, \delta)}{(\delta, \delta)} = (\gamma, \delta^\vee)$ $\forall \gamma, \delta \in E$, we have

no 4

$$(\alpha, \beta^\vee)(\beta, \alpha^\vee) = 4\cos^2\theta_{\alpha\beta} \in \{0, 1, 2, 3\}. \quad \forall \alpha, \beta \in \Phi$$

Actually we have, if we assume $\|\beta\| \geq \|\alpha\|$

① $(\alpha,\beta^\vee)(\beta,\alpha^\vee) \overset{= 4\cos^2\theta_{\alpha\beta}}{\in} \{0, 1, 2, 3\}$.

in particular, $(\alpha,\beta^\vee)$ and $(\beta,\alpha^\vee)$ have the same sign if $(\alpha,\beta) \neq 0$

② $$\frac{(\beta,\alpha^\vee)}{(\alpha,\beta^\vee)} = \frac{\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}{\frac{2(\alpha,\beta)}{(\beta,\beta)}} = \frac{(\beta,\beta)}{(\alpha,\alpha)} = \left(\frac{\|\beta\|}{\|\alpha\|}\right)^2 \geq 1.$$

equivalently, $$\frac{\|\beta\|}{\|\alpha\|} = \sqrt{\frac{(\beta,\alpha^\vee)}{(\alpha,\beta^\vee)}}$$

③ $(\alpha,\beta^\vee), (\beta,\alpha^\vee) \in \mathbb{Z}$ by (R4).

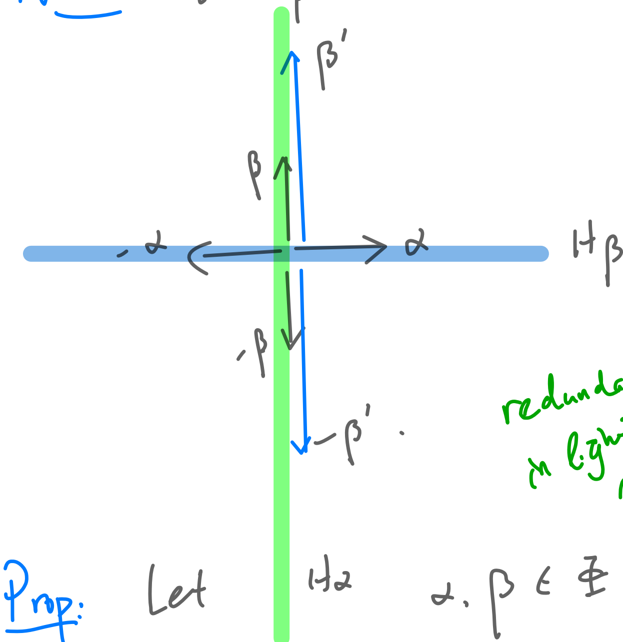
④ $(\alpha,\beta^\vee), (\beta,\alpha^\vee)$ are pos. if $\theta < \frac{\pi}{2}$ and neg if $\theta > \frac{\pi}{2}$.

①-④ leave few possibilities for $\left((\alpha,\beta^\vee), (\beta,\alpha^\vee)\right)$.

$\langle\alpha,\beta^\vee\rangle\langle\beta,\alpha^\vee\rangle = 4\cos^2\theta$	θ	$\langle\alpha,\beta^\vee\rangle$	$\langle\beta,\alpha^\vee\rangle$	$\dfrac{\lVert\beta\rVert}{\lVert\alpha\rVert}$
0	$\frac{\pi}{2}$	0	0	undetermined * possibly any positive value.
1	$\frac{\pi}{3}$	1	1	1
1	$\frac{2\pi}{3}$	-1	-1	1
2	$\frac{\pi}{4}$	1	2	$\sqrt{2}$
2	$\frac{3\pi}{4}$	-1	-2	$\sqrt{2}$
3	$\frac{\pi}{6}$	1	3	$\sqrt{3}$
3	$\frac{5\pi}{6}$	-1	-3	$\sqrt{3}$

We'll see these three examples.

Note: It's possible to have $\theta = \frac{\pi}{2}$ and $\frac{\|\beta\|}{\|\alpha\|} = r$ for any $r \in \mathbb{R}_{>0}$.



Pf: Compute $s_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta \in \Phi$

From the previous table, we have

$$(\alpha,\beta)^\vee = \begin{cases} 1 & \text{if } \theta_{\alpha\beta} < \frac{\pi}{2} \\ -1 & \text{if } \theta_{\alpha\beta} > \frac{\pi}{2}. \end{cases}$$

The prop. follows.

redundant assumption in light of $*$

Prop: Let $\alpha, \beta \in \Phi$ (with $\|\beta\| \geq \|\alpha\|$). see pf.

1) If $\theta_{\alpha\beta}$ is strictly obtuse (i.e. if $\theta_{\alpha\beta} > \frac{\pi}{2}$, i.e. if $(\alpha,\beta) < 0$), then

$$\alpha + \beta \in \Phi$$

2) If $\theta_{\alpha\beta}$ is strictly acute (i.e. if $\theta < \frac{\pi}{2}$, i.e. if $(\alpha,\beta) > 0$), then $\alpha - \beta \in \Phi$ (and hence $\underbrace{\beta - \alpha \in \Phi}_{*}$ as well).

2. Bases of a root system.

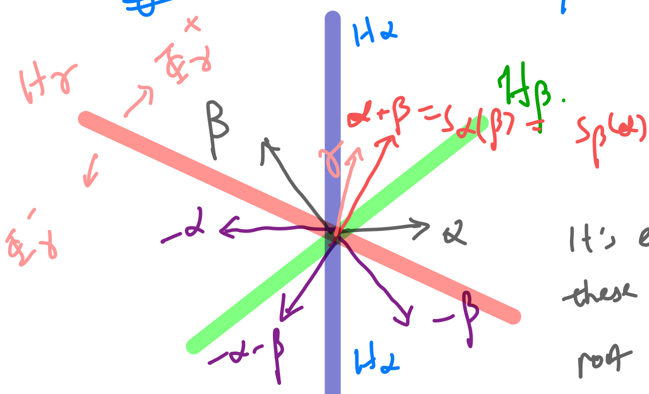
Def: A subset $\Delta \subseteq \Phi$ is called a base of (E, Φ) if:

(B1). Δ is a linear basis of E.

(B2). each root $\beta \in \Phi$ can be written in the form $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$

where either $k_\alpha \in \mathbb{Z}_{\geq 0}\ \forall \alpha \in \Delta$ or $k_\alpha \in \mathbb{Z}_{\leq 0}\ \forall \alpha \in \Delta$.

E.g. A Rank-2 root system. $E = \mathbb{R}^2$, α, β, $\theta_{\alpha\beta} = \frac{2\pi}{3}$. $\|\alpha\| = \|\beta\|$.



It's easy to check that these six vectors form a root system.

> **Thm:** Every root system has a base.

Pf of the theorem: If $\dim E = 1$, then $\Phi = \{\alpha, -\alpha\}$ for some $\alpha \in E$.

So $\Delta = \{\alpha\}$ works.

Assume $\dim E \geq 2$. Then $\exists\, \gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$. Let

$$\Phi_\gamma^+ = \{\alpha \in \Phi \mid (\alpha, \gamma) > 0\},$$

$$\Phi_\gamma^- = \{\alpha \in \Phi \mid (\alpha, \gamma) < 0\}.$$

Then $\Phi = \Phi_\gamma^+ \cup \Phi_\gamma^-$. Call $\alpha \in \Phi_\gamma^+$ <u>decomposable</u> if $\alpha = \alpha' + \alpha''$ for some $\alpha', \alpha'' \in \Phi_\gamma^+$ and <u>indecomposable</u> otherwise.

Let $\Delta_\gamma = \{\text{indecomposable elts in } \Phi_\gamma^+\}$.

Claim: Δ_γ is a base. (So Φ actually has multiple bases.)

Pf: next time.