Last time:

- Given any s.s. Lie algebra L and a <u>chosen</u> Cartan subalgebra H of L, we get a root system that is an abstract root system, via the Cartan decomposition (w.r.t. H)

$$(L, H) \longmapsto (E, \Phi)$$

- We'll classify s.s. Lie algebras (over $\mathbb{C}$) via root systems by establishing a bijection.

to be defined

$$\left\{ \begin{array}{c} \text{Iso classes of} \\ \text{s.s. Lie algebras} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \underline{\text{Iso}} \text{ classes of} \\ \text{(abstract) root systems} \end{array} \right\}.$$

To do this we need to

(1). Show that the process $(L, H) \mapsto (E, \Phi)$ gives a well-defined map $f: L \to (E, \Phi)$, with two choices of CSAs always yielding isomorphic root systems.

(2). Show that f is inj, i.e., two s.s. Lie algebras have iso root systems iff they are isomorphic. i.e., we can recover a s.s. Lie algebra from its root system.

(3). Show that f is surj. i.e., every root system can be realized as the root system of a s.s. Lie algebra.

(Involves construction. Lie algebras by generators and relations)

— Actually, f restricts to a bij

$$\{\text{simple Lie algebras}\} \longleftrightarrow \{\underline{\text{irreducible}} \text{ root systems}\}$$

to be defined

and it suffices to study irr. root systems.

— We'll classify irr. root systems by Dynkin diagrams.

· (E, Φ) + a chosen base Δ of Φ $\longrightarrow$ a graph

· different bases Δ, Δ' $\longrightarrow$ the same graph.

Dynkin diagram

· Fact: there's a bijection $\{\text{irr root systems}\} \longleftrightarrow \{\text{(connected) Dynkin diagrams}\}$.

We'll focus on root systems and their Dynkin diagrams first.

Classification of root systems. I.

Definitions / Notation / observations.

Let (E, Φ) be a root system.

real v.s. with pos. def. symm. real-valued bi-linear form $(\, , \,)$.

satisfies four axioms.

i.e. $E = \bigoplus_{\alpha \in \Phi} \mathbb{R}\alpha$

(R1) $|\Phi| < \infty$. $0 \notin \Phi$, $\mathrm{Span}(\Phi) = E$.

(R2) If $\alpha \in \Phi$, then the only mult. of α in Φ are α and $-\alpha$.

(R3) $\forall \beta, \alpha \in \Phi$. $\beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \alpha \in \Phi$.

(R4) $\forall \beta, \alpha \in \Phi$. $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Cartan int.

i.e. $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} = \beta(h_\alpha) \in \mathbb{Z}$

- $\forall \alpha \in \Phi$, denote the hyperplane perp. to α by H_α, i.e.,

$$H_\alpha = \{ v \in E \mid (v, \alpha) = 0 \}.$$

- $\forall \alpha \in \Phi$, we denote the reflection w.r.t. H_α by S_α. The reflection S_α may be characterized as the unique linear map s.t. $S_\alpha(\alpha) = -\alpha$ and $S_\alpha(v) = v \ \forall v \in H_\alpha$. ($E = H_\alpha \oplus \mathbb{R}\alpha$)

Prop: $$S_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \quad \forall \beta \in E.$$

Pf: Note that (1) S_α is linear, (2) $S_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$ and (3) $S_\alpha(v) = v - 0 \cdot \alpha = v \ \forall v \in H_\alpha$. $\square$

Def: The subgp of $GL(E)$ generated by the set

$$\{ s_\alpha : \alpha \in \Phi \}$$

is called the <u>Weyl group</u> of Φ $\left((E, \Phi) \right)$.

We usually denote it by W.

- $\forall \alpha \in \Phi$, we define the <u>coroot</u> of α to be the elt

$$\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)} \in E.$$

<u>Note:</u> $\alpha \in \Phi \not\Rightarrow \alpha^\vee \in \Phi$. Indeed, $\alpha \in \Phi$, $\alpha^\vee \in \Phi \Rightarrow (\alpha,\alpha) = 2$.

So, $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \alpha = \beta - (\beta, \alpha^\vee)\alpha$.

In (R4), $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = (\beta, \alpha^\vee)$.

Note: People often define $\underline{\langle \beta, \alpha \rangle} = (\beta, \alpha^\vee)$.

(EW & Hum included)

$\langle \cdot, \cdot \rangle$ is linear in the first coord but not in the second! And $\langle \cdot, \cdot \rangle$ is not symm.

Note: Now (R3) and (R4) have become

Strong constraint since $|\Phi| < \infty$ $\leftarrow$ (R3) $s_\alpha(\beta) \in \Phi$ $\forall \alpha, \beta \in \Phi$. i.e., Φ is closed under the action of the Weyl gp.

turns out to be a strong restriction as well $\leftarrow$ (R4) $(\beta, \alpha^\vee) \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$.

Ex. If Φ is a root system in E, then so is the set

$$\Phi^\vee = \{ \alpha^\vee : \alpha \in \Phi \}.$$

— Since E is an Euclidean space, length and angles make sense.

$\forall v \in E$, length $(v) = \|v\| = \sqrt{(v, v)}$ (so $(v,v) = \|v\|^2$.)

Also recall that the angle $\theta_{\alpha\beta}$ between $\alpha, \beta \in E$ is the unique angle in $[0, \pi]$ s.t.

$$(\alpha, \beta) = \|\alpha\| \cdot \|\beta\| \cdot \cos\theta_{\alpha\beta}$$

Prop: $(\beta, \alpha^\vee)(\alpha, \beta^\vee) = 4\cos^2\theta_{\alpha\beta} \in \{0, 1, 2, 3, 4\}$.

only when $\beta = \pm\alpha$

$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle$

Cor: $(\beta, \alpha^\vee) \in \{0, \pm 1, \pm 2, \pm 3\}$ $\forall \alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$.

Pf: $(\beta, \alpha^\vee)(\alpha, \beta^\vee) = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \cdot \dfrac{2(\alpha, \beta)}{(\beta, \beta)} = \dfrac{4\|\alpha\|^2\|\beta\|^2\cos^2\theta_{\alpha\beta}}{\|\alpha\|^2\|\beta\|^2} = 4\cos^2\theta_{\alpha\beta}$.

Ex. Recall that the set $\Phi = \left\{ e_i - e_j \;\middle|\; \begin{matrix} 1 \le i, j \le n \\ i \ne j \end{matrix} \right\}$ forms a root system inside $E = \{ (a_1, \ldots, a_n) \in \mathbb{R}^n : a_1 + \cdots + a_n = 0 \}$.

- Compute S_α $\forall \alpha \in \Phi$ and then deduce what the Weyl gp is.

- Compute $(\beta, \alpha^\vee)$ $\forall \alpha, \beta \in \Phi$.