Progress so far: L s.s. $\quad L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$

I. $0 \neq x_\alpha \in L_\alpha$, $\alpha \in \Phi$ $\xrightarrow{\text{"orthogonality properties"}}$ $sl_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$

$$h_\alpha = \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha = \frac{2}{\alpha(t_\alpha)} t_\alpha$$

$sl_\alpha \circlearrowright L$

restrict $\swarrow$ $\qquad$ $\searrow$ restrict

$sl_\alpha \circlearrowright H \oplus \bigoplus_{c \in \mathbb{C}^*} L_{c\alpha}$

$\Downarrow$

$\dim L_\alpha = 1$

$c\alpha \in \Phi \Rightarrow c = \pm 1$

"integrality properties"

$sl_\alpha \circlearrowright \bigoplus L_{\beta + i\alpha}$, $\beta \in \Phi$, $\beta \neq \pm\alpha$

$\Downarrow$

- α root string through β.

  $\beta - r\alpha, \ldots\ldots, \beta + q\alpha \in \Phi$

- $\beta(h_\alpha) = r - q \in \mathbb{Z}$. Cartan integer.
- $\beta - \beta(h_\alpha)\alpha \in \Phi$

II. An inner product space inside H^*.

The "inner product" / bilinear form : $(\alpha,\beta) := K(t_\alpha, t_\beta) \quad \forall \alpha,\beta \in H^*$

Easy facts:

(0) $h_\alpha = \frac{2}{(\alpha,\alpha)} t_\alpha$, so $t_\alpha = \frac{(\alpha,\alpha)}{2} h_\alpha$.

(1) $\beta(h_\alpha) = K\left(t_\beta, \frac{2}{K(t_\alpha,t_\alpha)} t_\alpha\right) = \frac{2}{(\alpha,\alpha)} \cdot (\beta,\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

(2) $K(h_\alpha, h_\alpha) \overset{(0)}{=} K\left(\frac{2}{(\alpha,\alpha)} t_\alpha, \frac{2}{(\alpha,\alpha)} t_\alpha\right) = \frac{4}{(\alpha,\alpha)^2} (\alpha,\alpha) = \frac{4}{(\alpha,\alpha)}$

(3) $K(h_\alpha, h_\beta) = \sum_{\gamma\in\Phi} \gamma(h_\alpha)\gamma(h_\beta) \in \mathbb{Z}$.

(4) $\frac{4}{(\alpha,\alpha)} = K(h_\alpha, h_\alpha) \in \mathbb{Z}$ by (2) and (3).

The inner product space

Prop 1. We have $(\alpha, \beta) \in \mathbb{Q}$ $\forall\ \alpha, \beta \in \Phi$.

pf: $(\alpha,\beta) = \kappa(t_\alpha, t_\beta) = \kappa\left(\frac{(\alpha,\alpha)}{2} h_\alpha, \frac{(\beta,\beta)}{4} h_\beta \right)$

$$= \underbrace{\frac{(\alpha,\alpha)}{2}}_{\in\,\mathbb{Q}} \underbrace{\frac{(\beta,\beta)}{2}}_{\in\,\mathbb{Q}} \cdot \underbrace{\kappa(h_\alpha, h_\beta)}_{\in\,\mathbb{Z}} \in \mathbb{Q}. \quad \square$$

Next. Recall that Φ spans H^*, so a subset $\{\alpha_1, \ldots, \alpha_\ell\} \subseteq \Phi$ forms a basis for H^*.

Def: Let $E = \bigoplus_{i=1}^{\ell} \mathbb{R}\alpha_i$. (This will be our i.p. space.)

Prop 2. We have $\Phi \subseteq E$, i.e., if $\beta \in \Phi$ has decomp $\beta = c_1\alpha_1 + \cdots + c_\ell \alpha_\ell$, then $c_1, \ldots, c_\ell \in \mathbb{R}$. (actually we'll show $c_i \in \mathbb{Q}$ $\forall i$).

Pf: $\forall j \in [l] := \{1, 2, \ldots, l\}$.

$$(\beta, \alpha_j) = \sum_{i=1}^{l} c_i (\alpha_i, \alpha_j)$$

So
$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^{l} \frac{2 c_i (\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^{l} \frac{2 c_i (\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)} . \quad (*)$$

Recall that $\quad LHS = \beta(h_{\alpha_j}) \in \mathbb{Z}$.

Now we can rewrite $(*)$ as a matrix equation.

$$\begin{array}{r} \\ \\ \\ \\ j\text{th row} \\ \\ \end{array} \overset{A}{\begin{bmatrix} \frac{2(\alpha_1, \alpha_1)}{(\alpha_1, \alpha_1)} & \cdots \quad \cdots & \frac{2(\alpha_1, \alpha_l)}{(\alpha_1, \alpha_1)} \\ & \vdots & \\ \frac{2(\alpha_j, \alpha_1)}{(\alpha_j, \alpha_j)} & \cdots & \frac{2(\alpha_j, \alpha_l)}{(\alpha_j, \alpha_j)} \\ \cdot & \cdots & \cdot \end{bmatrix}} \overset{x}{\begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_l \end{bmatrix}} = \overset{b}{\begin{bmatrix} \beta(h_{\alpha_1}) \\ \vdots \\ \vdots \\ \beta(h_{\alpha_l}) \end{bmatrix}} .$$

Note: - $b \in \mathbb{Z}^{\ell}$.

- A is invertible since it can be obtained from the Gram matrix of the Killing form by scaling rows and the Killing form is nondegenerate.

- A has rational entries since $\frac{2(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)} = \alpha_i(h_{\alpha_j}) \in \mathbb{Z}$.
so A^{-1} has rational entries.

So $x = A^{-1}b \in \mathbb{Q}^{\ell}$. ie. $c_i \in \mathbb{Q}$ $\forall i \in [\ell]$. $\square$

So, $(\ ,\ )$ restricts to a real-valued symm. bilinear form on E.

Prop. $(\ ,\ )_E$ is positive definite and hence an inner product.

Pf: Let $\theta \in E$. Then

$$(\theta, \theta) = \kappa(t_\theta, t_\theta) = \sum_{\beta \in \Phi} \beta(t_\theta)\beta(t_\theta) = \sum_{\beta \in \Phi} (\underbrace{t_\beta, t_\theta}_{\in \mathbb{Q}})^2 .$$

It follows that $(\theta, \theta) \geq 0$, with equality holding iff

$$(t_\beta, t_\theta) = 0 \ \forall \beta \in \Phi .$$

Since Φ spans H^*, $(t_\beta, t_\theta) = \beta(t_\theta) = 0 \ \forall \beta \in \Phi \Rightarrow t_\theta = 0 \Rightarrow \theta = 0$. $\square$

We have now proved that the pair (E, Φ) satisfies the following axioms for an "abstract root system":

Def. A <u>root system</u> is a pair (E, Φ) where E is an inner

(Hum 9.2) product space and Φ is a subset of E satisfying the following

conditions.

(R1) Φ is finite ✓, spans E ✓, and $0 \notin \Phi$ ✓.

(R2) If $\alpha \in \Phi$, then the only multiples of α in Φ are α and $-\alpha$. ✓
(and $-\alpha \in \Phi$)

(R4). $\forall \alpha, \beta \in \Phi$, $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. ✓

for our root system from L.

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \beta(h_\alpha).$$

(R3). $\forall \alpha, \beta \in \Phi$, $\beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$ ✓

Ex. (HW) Work out t_α, (α,β), $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$, and $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$.

for all $\alpha, \beta \in \Phi$ in sl_n.

$$\Phi = \{ \varepsilon_i - \varepsilon_j \mid 1 \le i,j \le n,\ i \ne j \}.$$

Ex: Consider the usual Euclidean space $E = \mathbb{R}^n = \langle \underbrace{e_1, \ldots, e_n}_{\text{usual standard basis}} \rangle$.

Show that $\Phi = \{ e_i - e_j \mid 1 \le i,j \le n,\ i \ne j \}$ forms a root system.

Work out (α,β), $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$, $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ for all $\alpha, \beta \in \Phi$.

Ex: How do these two root systems relate to each other?