Last time: Working towards integrality properties of Φ.

(3) $h_\alpha = c\alpha(h_\alpha)$
$= 2c$

Key: look at $sl_2 \circlearrowright M = H \oplus \bigoplus_{c \in \mathbb{C}^*} L_{c\alpha}$

Recall: $H = \ker\alpha \oplus F h_\alpha$. So if we pick $x_\alpha \in L_\alpha$, $0 \neq x_\alpha$.

then $M = \underset{sl_2 = 0}{\ker\alpha} \oplus \left(F h_\alpha \oplus F x_\alpha \oplus F y_\alpha \right) \oplus \left(\begin{matrix} \text{other things} \\ \text{in } L_{c\alpha},\ c \neq 0 \end{matrix} \right)$

"$\in L_{c\alpha}$"

Note: (1) sl_α acts on $\ker\alpha$ trivially: take $h \in \ker\alpha$.

$[x_\alpha, h] = -[h, x_\alpha] = -\alpha(h)\, x_\alpha = 0 \cdot x_\alpha = 0$

$[y_\alpha, h] = +\alpha(h)\, y_\alpha = 0 \cdot y_\alpha = 0$

$[h_\alpha, h] = 0$

(2) On $F h_\alpha \oplus F x_\alpha \oplus F y_\alpha$, sl_α acts by the adj rep.

So, "$\oplus L_{c\alpha}$" in the "other things" contains no wt space w/ wt 0.

$\Rightarrow$ the same "$\oplus L_{c\alpha}$" has no even wts by sl_2-rep theory.

in particular, "$\oplus L_{c\alpha}$" contains no contribution w/ $c=2$.

Point: $\underline{\beta \in \Phi \Rightarrow 2\beta \notin \Phi}$. "twice a root is never a root".

It follows that $\frac{1}{2}\alpha \notin \Phi$, so "$\oplus L_{c\alpha}$" has no contribution w/ $c=\frac{1}{2}$. But then "$\oplus L_{c\alpha}$" has no wt space of wt 1. So $\oplus L_{c\alpha}=0$ since it contains neither 0- nor 1- wt spaces. So, $M = H \oplus sl_2$.

It follows that

(a) $\dim L_\alpha = 1 \quad \forall \alpha \in \Phi$

(b) $\alpha \in \Phi, \quad c\alpha \in \Phi, \quad c \in \mathbb{C}^* \Rightarrow c = 1$ or $c = -1$.

Next, consider the sl_α action on the space

$$K = \bigoplus_{\substack{i \in \mathbb{Z} \\ \beta + i\alpha \in \Phi}} L_{\beta + i\alpha} \quad \text{for a root } \beta \in \Phi, \ \beta \neq \pm\alpha.$$

By the above discussion, $\dim L_{\beta+i\alpha} = 1$, and $\beta + i\alpha \neq 0$.

$\forall x \in L_{\beta+i\alpha}$, $\quad [h_\alpha, x] = (\beta + i\alpha)(h_\alpha) \cdot x = [\beta(h_\alpha) + 2i]\, x$.

The wt $\beta(h_\alpha) + 2i$ is an integer, so $\underbrace{\beta(h_\alpha)}_{\text{Cartan integer.}}$ is an integer.

Moreover, all the wts of the form $\beta(h_\alpha) + 2i$ have the parity, and they either contain exactly one occurrence of 0 or 1, so (by Weyl's C.R. Thm and $\mathfrak{sl}_2$-rep theory).

$K = \bigoplus L_{\beta + i\alpha}$ must be an irreducible $\mathfrak{sl}_\alpha$-module.

In particular, we can find largest integers r, q s.t. $\beta(h_\alpha) - 2r$, $\beta(h_\alpha) + 2q$ are the lowest and highest wt in L. Note that $\beta(h_\alpha) + 2q = -(\beta(h_\alpha) - 2r)$, or, equiv. $\beta(h_\alpha) = r - q$, and it follows that all elts $\beta - r\alpha, \beta - (r-1)\alpha, \ldots, \beta + (q-1)\alpha, \beta + q\alpha$ all appear in Φ.

In particular, $\beta - \beta(h_\alpha)\alpha = \beta - (r-q)\alpha$

$= \beta - r\alpha + q\alpha$

must appear "in the middle" of the root string. i.e.,

$$\beta - \beta(h_\alpha)\alpha \in \Phi.$$

We have now proved.

(c) $\forall \alpha, \beta \in \Phi$, $\beta(h_\alpha) \in \mathbb{Z}$, and $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(e) $\forall \alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$, $\exists$ an "α-root string through β"

$\beta - r\alpha, \ldots, \beta + q\alpha$ in Φ, with $\beta(h_\alpha) = r - q$.

It remains to prove (d) and (f).

(d). $\alpha, \beta \in \Phi$, $\alpha+\beta \in \Phi \Rightarrow [L_\alpha, L_\beta] = L_{\alpha+\beta}$

Pf: Consider

$$h_\alpha \in sl_\alpha \quad \begin{tikzcd} L_\beta \arrow[loop left, "\beta(h_\alpha)"] \arrow[d, "e"] \\ L_{\alpha+\beta} \arrow[loop left, "\beta(h_\alpha)+2"] \end{tikzcd}$$

$L_{\alpha+\beta} \neq 0$

$\Downarrow$

the h.w. in $K = \oplus L_{\beta+i\alpha}$ is not in L_β.

$\Downarrow$

for $0 \neq v \in L_\beta$, $e_\alpha \cdot v$ is a nonzero vector in $L_{\alpha+\beta}$.

Done since $\dim L_{\alpha+\beta} = 1$.

(f). L is generated by $\{L_\alpha : \alpha \in \Phi\}$.

Pf: follows from (d).

$\square$.

Next: An inner product (need real-valued, positive-bilinear form on $\mathbb{R}$-vect. space)

- Recall the def of t_α for $\alpha \in H^*$: it's the unique elt $t_\alpha \in H$. $K(t_\alpha, -) = \alpha$.

i.e. $\alpha(h) = K(t_\alpha, h)$.

Define a bilinear form $(\, , \,)$ on H by

bilinear form: $\longrightarrow$ $$(\alpha, \beta) = K(t_\alpha, t_\beta) \quad \forall \alpha, \beta \in H$$

$$K(t_\alpha, t_\beta) = \alpha(t_\beta) = \beta(t_\alpha).$$

- Also recall that $h_\alpha = [x_\alpha, y_\alpha] = K(x_\alpha, y_\alpha) t_\alpha = \dfrac{2}{K(t_\alpha, t_\alpha)} t_\alpha$

E.x. on $(\, , \,)$: (1) $\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \kappa\left(t_\beta, \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha\right)$

(2) $\kappa(h_\alpha, h_\alpha) = \frac{4}{\kappa(t_\alpha, t_\alpha)}$ $\qquad = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$

important equality for later: $\boxed{\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}}$

Immediate goal: $(\alpha,\beta) \in \mathbb{Q}$ $\forall \alpha,\beta \in \Phi$. $\leftarrow$ "$(\, , \,)$ is real-valued".

Prop: $\forall \alpha,\beta \in \Phi$, we have $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $(\alpha,\beta) \in \mathbb{Q}$.

Pf: (3) $\kappa(h_\alpha, h_\beta) \overset{\text{def}}{=} \operatorname{tr}(\operatorname{ad} h_\alpha \operatorname{ad} h_\beta) \overset{L = H \oplus \bigoplus_{\gamma\in\Phi} L_\gamma}{=} \sum_{\gamma\in\Phi} \overset{\mathbb{Z}}{\gamma(h_\alpha)}\,\overset{\mathbb{Z}}{\gamma(h_\beta)} \in \mathbb{Z}$.

$$(\alpha,\beta) = \kappa(t_\alpha, t_\beta) = \kappa\left(\frac{\kappa(t_\alpha,t_\alpha)}{2} h_\alpha, \frac{\kappa(t_\beta,t_\beta)}{2} h_\beta\right)$$

$$= \underbrace{\tfrac{1}{4}\kappa(t_\alpha,t_\alpha)\kappa(t_\beta,t_\beta)}_{\text{rational by (2) and (3)}} \kappa(h_\alpha, h_\beta) \in \mathbb{Q}. \qquad \square$$