Goal: H Cartan subalgebra of s.s. L $\longrightarrow$ $L = \underset{\overset{\|}{C_L(H)=H}}{L_0} \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$

understand Φ and the spaces L_α, $\alpha \in \Phi$.

Last time: – $K|_H$ is nondegenerate $\to$ get identification $H^* \to H$

$\alpha \in H^* \longmapsto t_\alpha$, the unique elt in H s.t. $\boxed{\alpha(h) = K(t_\alpha, h)}\ \forall h \in H$.

haven't shown dim $L_\alpha = 1$ $\forall \alpha \in \Phi$

– If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

– $\forall x_\alpha \in \Phi$, $x_\alpha \neq 0$, $\exists\ y_\alpha \in L_{-\alpha}$ and $h_\alpha \in H$ s.t. $\boxed{\alpha(h_\alpha) = 2}$

$sl_\alpha := \langle x_\alpha, y_\alpha, h_\alpha \rangle \cong sl_2$

– Moreover, $\boxed{h_\alpha = \dfrac{2}{K(t_\alpha, t_\alpha)} t_\alpha} = \dfrac{2}{\alpha(t_\alpha)} t_\alpha$

Note: $\alpha(h_\alpha) = K(t_\alpha, h_\alpha) = \dfrac{2}{K(t_\alpha, t_\alpha)} K(t_\alpha, t_\alpha) = 2$.

— **Note:** Since $\alpha: H \to \mathbb{C}$ is linear, dim considerations imply that $H = \ker\alpha \oplus \langle h_\alpha \rangle$

Today. Examine the action $sl_\alpha \overset{ad}{\circlearrowright} L$ to get more information.

We'll need Weyl's Complete Reducibility Thm.

Weyl's Thm. Every f.d. rep of a $\begin{matrix}\text{f.d.}\\ \text{s.s.}\end{matrix}$ Lie algebra is a direct sum of simple reps.

Aside: Clebsch-Gordan rule for sl_2.

Recall that $\forall d \in \mathbb{Z}_{\geq 0}$, $\exists!$ $\mathfrak{sl}_2$-irrep V_d with dim $d+1$.

and we can find a basis $\{v_d, v_{d-2}, \dots, v_{-d}\}$ of V_d s.t.

$$\begin{tikzcd}
0 & v_{-d} \arrow[l, bend left] \arrow[r, bend left] \arrow[loop above, "h=-d"] & v_{-d+2} \arrow[l, bend left] \arrow[r, bend left] \arrow[loop above, "h=-d+2"] & \cdots & v_{d-4} \arrow[l, bend left] \arrow[r, bend left] \arrow[loop above, "h=d-4"] & v_{d-2} \arrow[l, bend left] \arrow[r, bend left] \arrow[loop above, "h=d-2"] & v_d \arrow[l, bend left] \arrow[r, bend left] \arrow[loop above, "h=d"] & 0
\end{tikzcd}$$

(Earlier we got the existence of the h.w. vector and the eigenbasis from "Pigeonhole principle". but now we can use A.J.D to get the eigenbasis directly.)

<u>Q:</u> How does $V_4 \otimes V_3$ decompose into irreducibles?

<u>A:</u> $V_4 \otimes V_3 = V_7 \oplus V_5 \oplus V_3 \oplus V_1$; enough to consider wts $\left(\overset{\alpha \in H^* \mapsto \alpha(h)}{\longleftrightarrow} \mathbb{C} \right)$.

Here's how wts determine the decomposition :

recall that in $V \otimes W$, V, W L-modules, we have

$$x.(v\otimes w) = x.v \otimes w + v \otimes x.w.$$

So if $v \in V_\alpha$, $w \in W_\beta$, then $v\otimes w \in (V\otimes W)_{\alpha+\beta}$.

$$\Downarrow \qquad\qquad\qquad\qquad \Uparrow$$

$\forall x \in H$, $x.v = \alpha(x)\, v$, $x.w = \beta(x)\, w$. $\Rightarrow$ $x.(v\otimes w) = (\alpha+\beta)(x)\,(v\otimes w)$.

So take the basis $\{v_4, v_2, v_0, v_{-2}, v_{-4}\}$ of $V = V_4$ and

the basis $\{w_3, w_1, w_{-1}, w_{-3}\}$. of $W = V_3$.

Then $\{ v_i \otimes w_j$, suitable $i, j\}$ form a eigenbasis for $H = \langle h \rangle$.

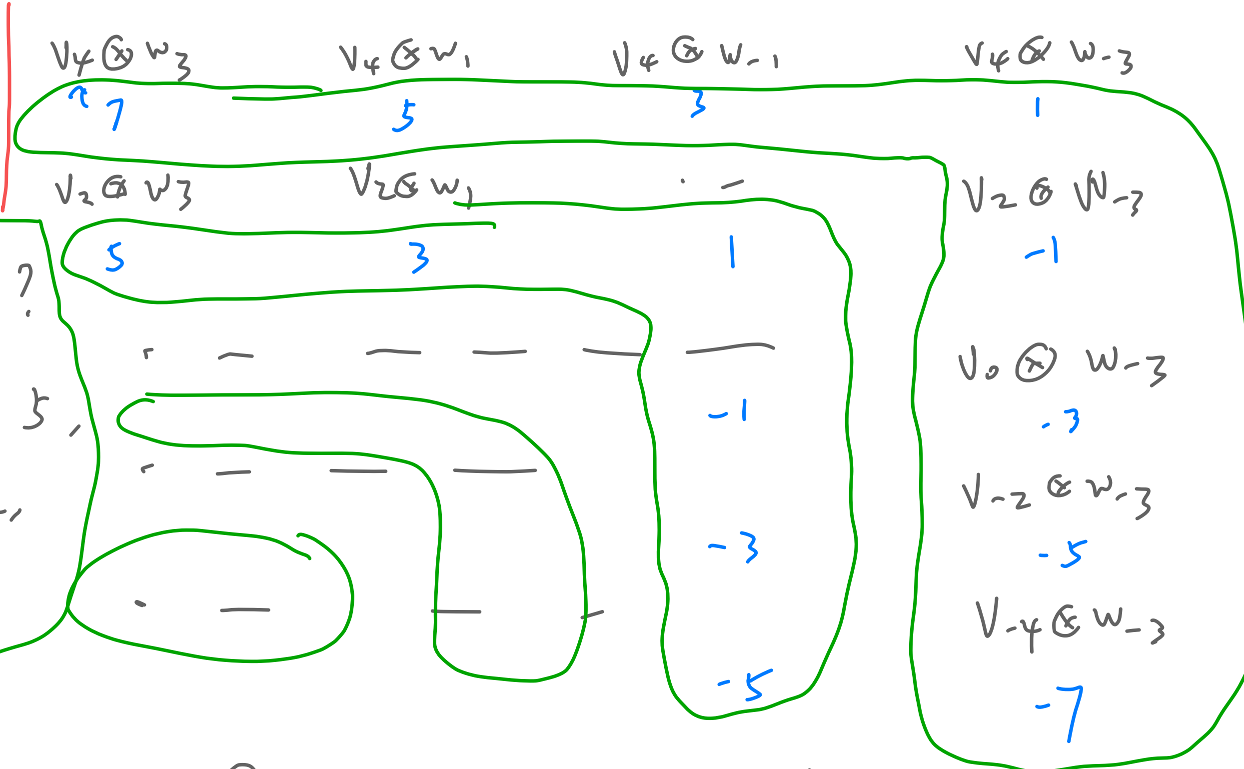
Point: "Wts control f.d. reps of s.s. Lie algebras."

$V_4 \otimes V_3 = V_7 \oplus\ ?$

largest wt in ? is 5, and 5 appears once,

$\downarrow$

V_5.



Say $V_4 \otimes V_3 = \bigoplus_{n \in S} V_n$, and compare the wt multiset.

the largest wt is 7, and it appears once on the left,

so $\max(S) = 7$ and 7 appears once in S. Remove V_7 and repeat ...

Back to the sl_2 actions.

Results. (Hum 8.4) Let $\alpha \in \Phi$ and consider $sl_\alpha := \langle x_\alpha, y_\alpha, h_\alpha \rangle$. We have:

(a) $\dim L_\alpha = 1$. In particular $sl_\alpha = L_\alpha \oplus L_{-\alpha} \oplus \underset{\langle h_\alpha \rangle}{\underset{\shortparallel}{H_\alpha}}$,

and for any $0 \neq x_\alpha \in L_\alpha$, $\exists!\ y_\alpha \in L_{-\alpha}$ s.t. $[x_\alpha, y_\alpha] = h_\alpha$.

(b) The only multiples of α in $\Phi \subsetneq H^*$ are α and $-\alpha$.

(c) If $\beta \in \Phi$, then $\underbrace{\beta(h_\alpha)}_{\text{a Cartan integer}} \in \mathbb{Z}$. and $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(d) If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

(e). If $\beta \in \Phi$ and $\beta \neq \pm\alpha$. Let r, q be the largest integers s.t.

$\beta - r\alpha, \beta + q\alpha \in \Phi$, Then $\beta + i\alpha \in \Phi \ \forall -r \le i \le q$, and

$\beta(h_\alpha) = r - q$. (e.g. $\beta - \alpha$, β, $\beta+\alpha$, $\beta+2\alpha$, $\beta+3\alpha$.)

(f). L is generated as a Lie algebra by the root spaces $L_\alpha, \alpha \in \Phi$.

Pf: Consider $M := \underbrace{H}_{h_\alpha \text{ acts as } 0} \oplus \bigoplus_{\substack{c \in \mathbb{C} \\ c \neq 0}} \underbrace{L_{c\alpha}}$ h_α acts as $c\alpha(h_\alpha) = 2c$ on $L_{c\alpha}$.

Recall that $H = \ker\alpha \oplus \langle h_\alpha \rangle$, so

(a).(b) hold $\Leftrightarrow \square = 0$.

$$M = (\ker\alpha \oplus \langle h_\alpha \rangle) \oplus \underbrace{\langle x_\alpha \rangle}_{\hat{L}_\alpha} \oplus \underbrace{\langle y_\alpha \rangle}_{\hat{L}_{-\alpha}} \oplus \boxed{\text{other parts.}}$$

one copy of sl_2.